Induction Machines

Three phase Induction Motor

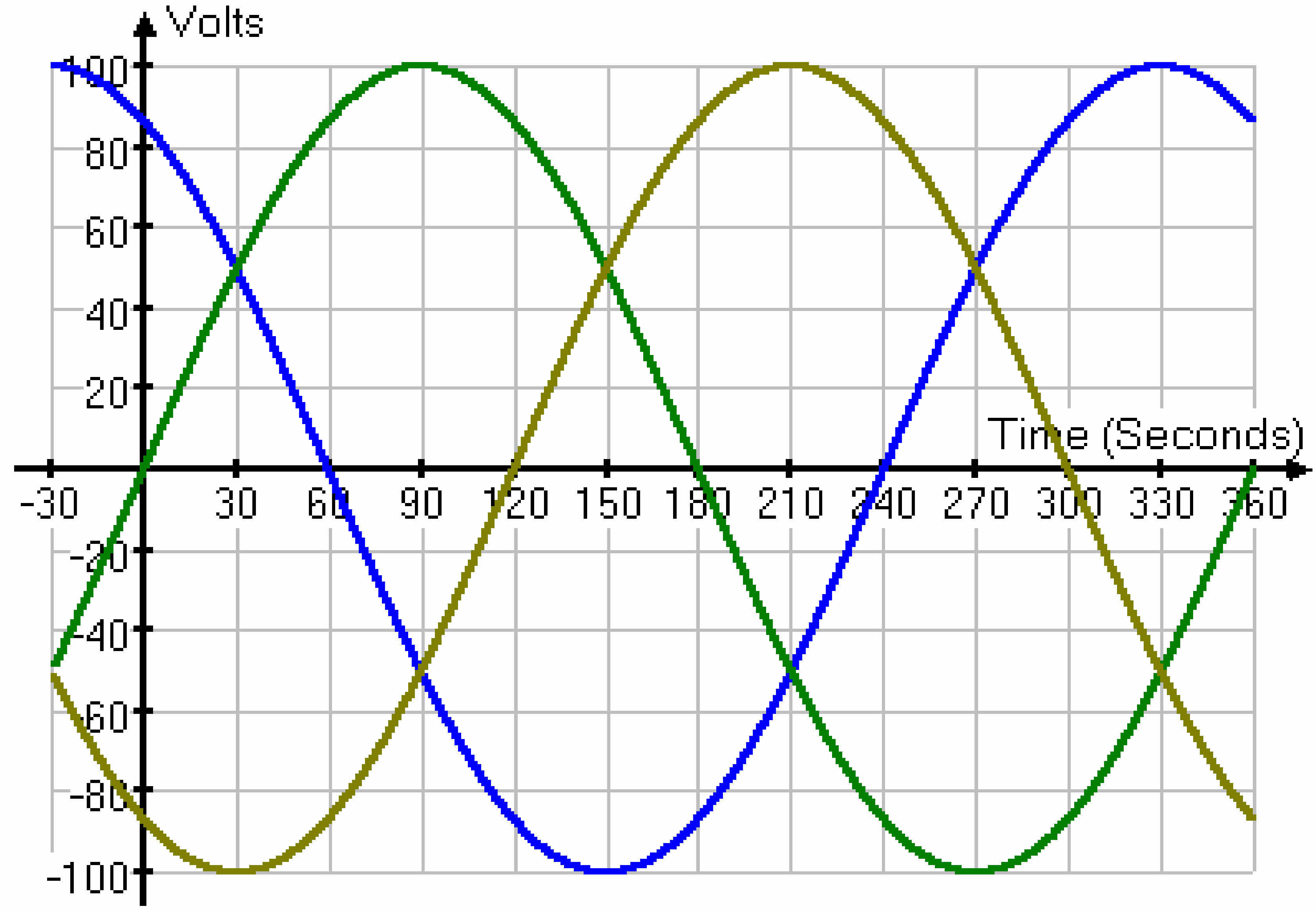
- Advantages and disadvantages of 3-phase induction motor
- Production of rotating magnetic field
- Definition of Slip
- Single phase equivalent circuit
- Torque equation
- Slip at maximun torque
- Effect of external rotor resistance

Advantages and disadvantages of 3 phase induction motor

- **Advantages:**
- Induction motor is cheap and robust
- The motor is driven by the rotational magnetic field produced by 3 phase currents, hence no commutator or blush is required
- Maintenance is relatively easy and at low cost
- **Disadvantages:**
- Induction motor has low inherent starting torque
- Difficult to control the speed of induction motor

Production of magnetic field by three phase voltages

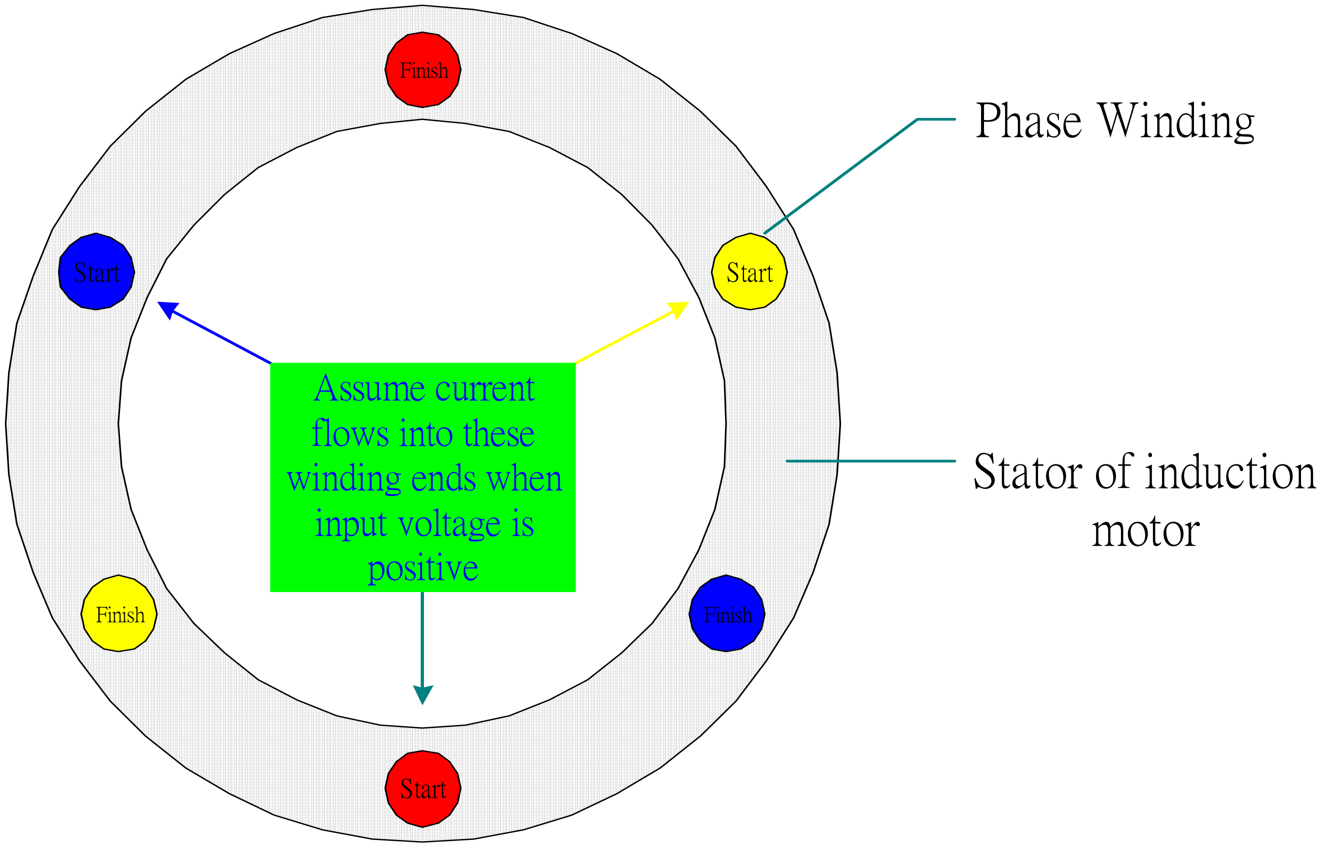
- Three phase voltages has a phase displacement of 120 degrees, hence at any instant of time there is a different values of three phase voltages in Volts. These voltages can be positive or negative since the three phase voltage is sinusoidal in waveform
- Let us examine one complete cycle of a very low frequency three phase voltage (period T = 360 seconds), at an interval of 30 electrical degrees, and we tabulate the three phase voltages in the following table:


Volts
100
80
60
40
20
-20
-40
-60
-80
-100
Time (Seconds)
-30
30
60
90
120
150
180
210
240
270
300
330
360

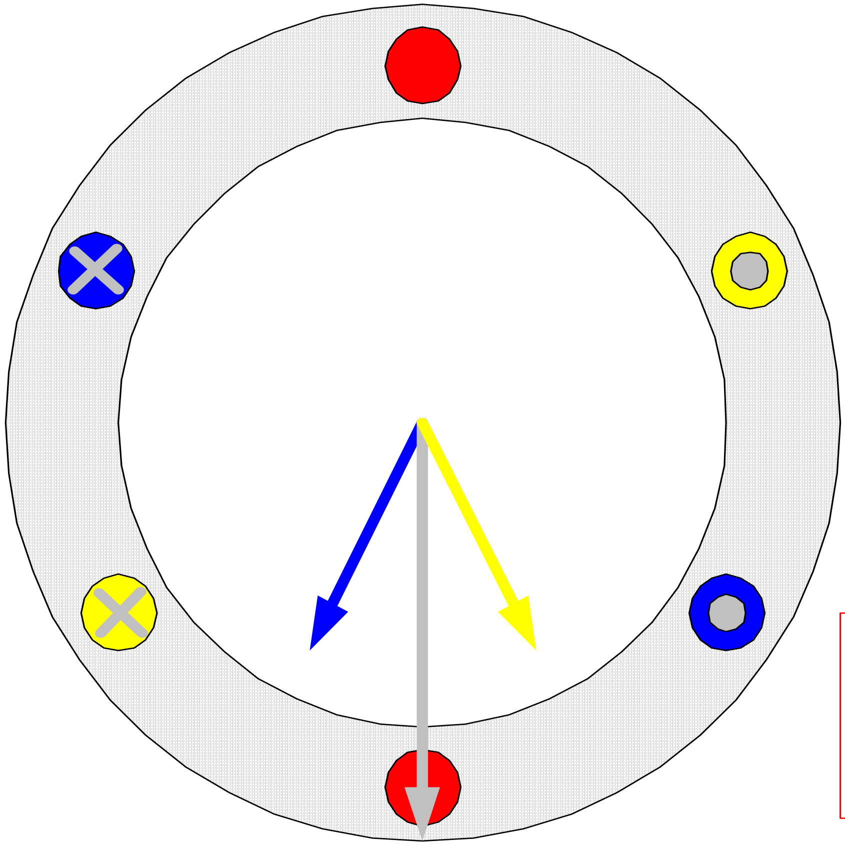
Instantaneous Values of three phase Voltages

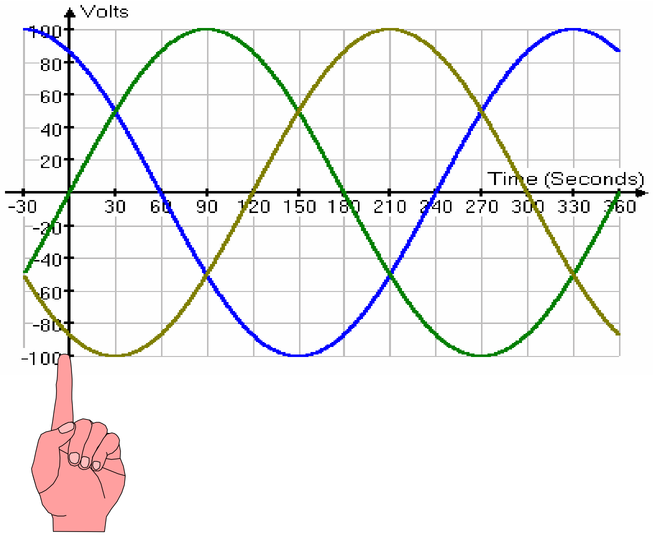
		3 Phase Voltage (Volts)		
Time	**angle**	*Red Phase*	*Yellow phase*	*Blue Phase*
0	0	0	-86.6	86.6
1	30	50	-100	50
2	60	86.6	-86.6	0
3	90	100	-50	-50
4	120	86.6	0	-86.6
5	150	50	50	-100
6	180	0	86.6	-86.6
7	210	-50	100	-50
8	240	-86.6	86.6	0
9	270	-100	50	50
10	300	-86.6	0	86.6
11	330	-50	-50	100
12	360	0	-86.6	86.6

Magnetic Field produced by three phase currents when 3 phase voltage is input to the stator of a 2 pole induction motor



Magnetic Field produced in a 2 pole induction motor

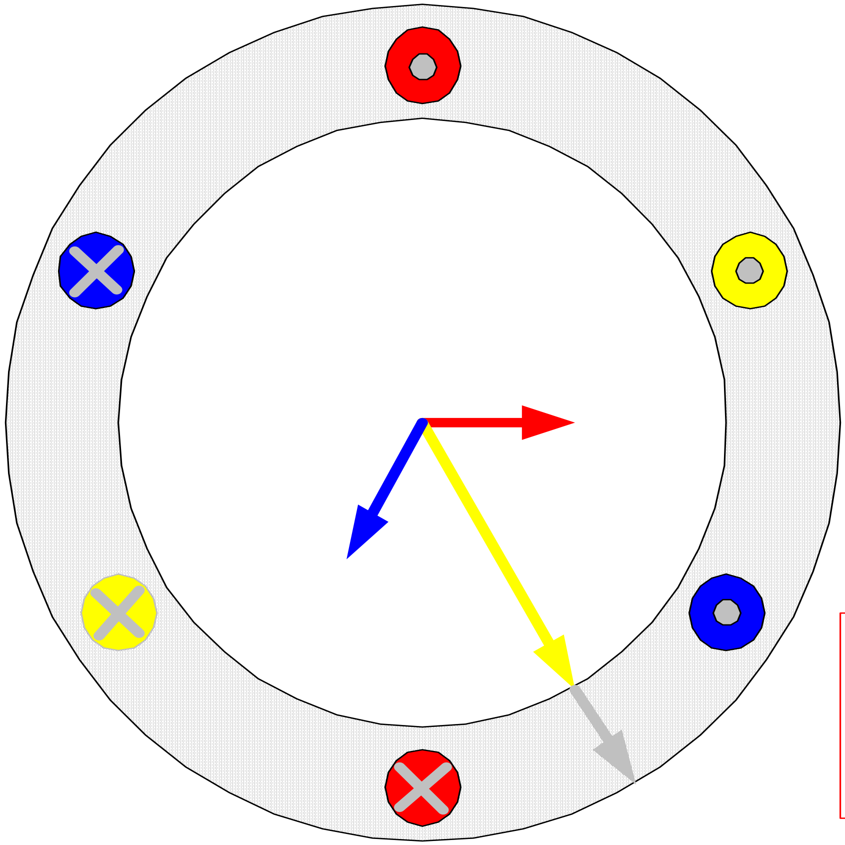


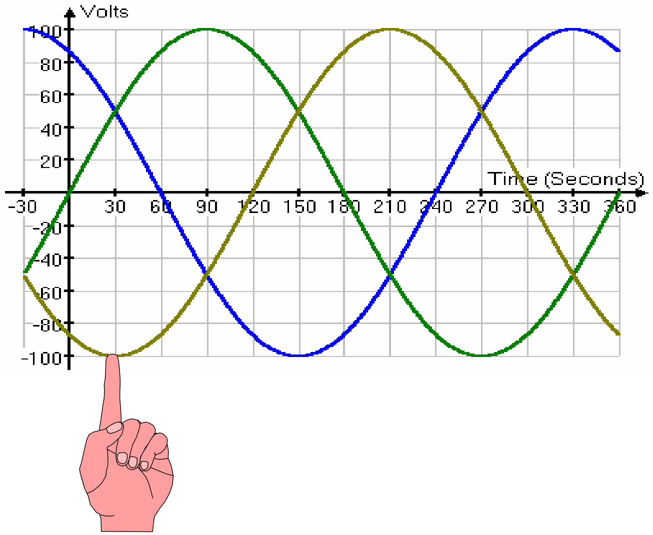


Time : t = 0

Red phase = 0 V
Yellow Phase = -86.6 V
Blue Phase = +86.6 V

Magnetic Field produced in a 2 pole induction motor

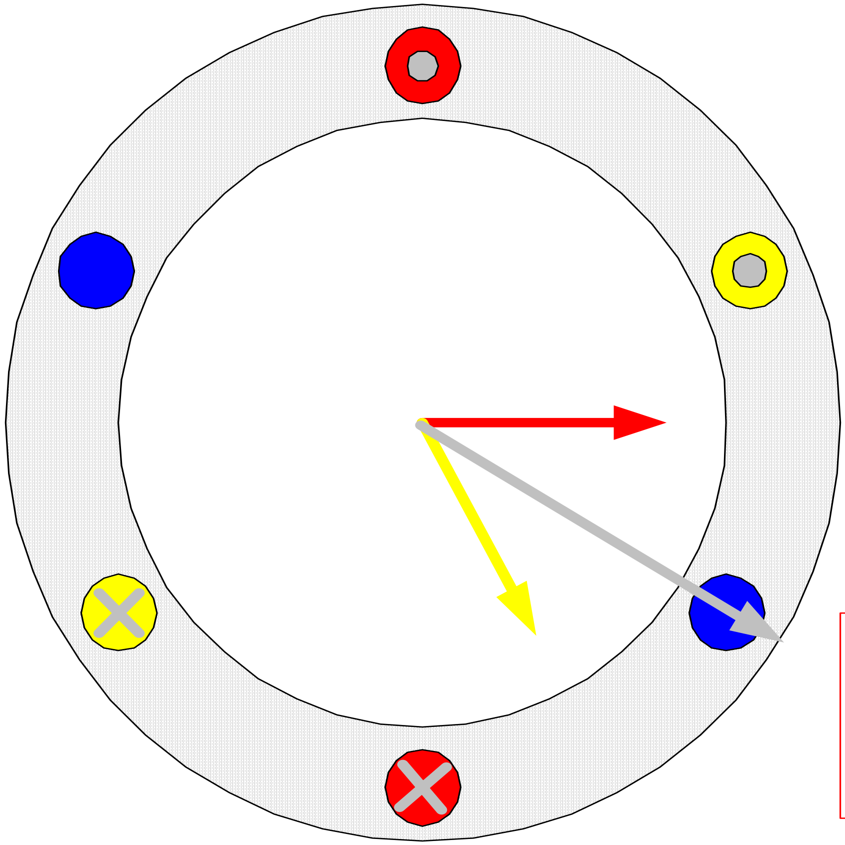


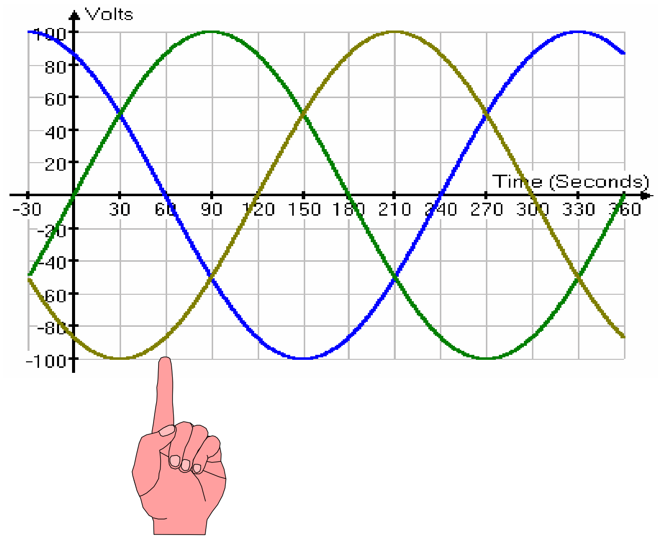


Time : t = 30 Sec

Red phase = 50 V
Yellow Phase = - 100 V
Blue Phase = + 50 V

Magnetic Field produced in a 2 pole induction motor

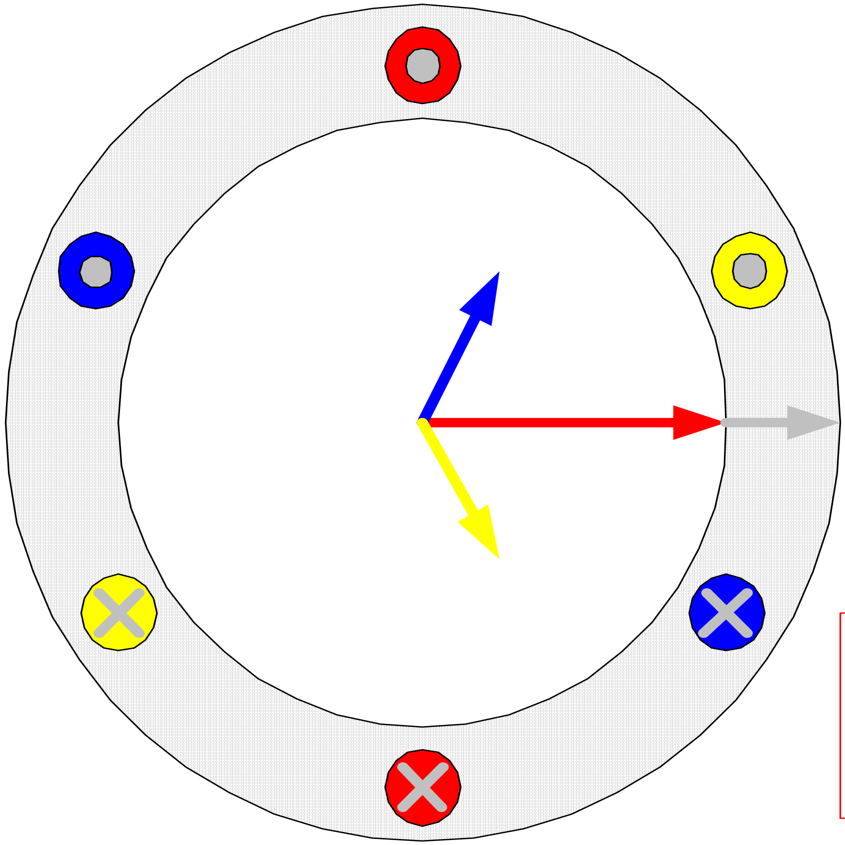


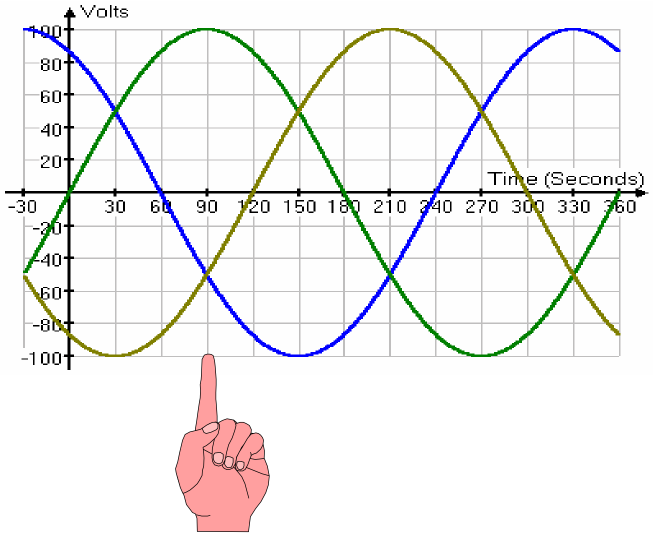


Time : t = 60 sec

Red phase = +86.6 V
Yellow Phase = -86.6 V
Blue Phase = 0 V

Magnetic Field produced in a 2 pole induction motor

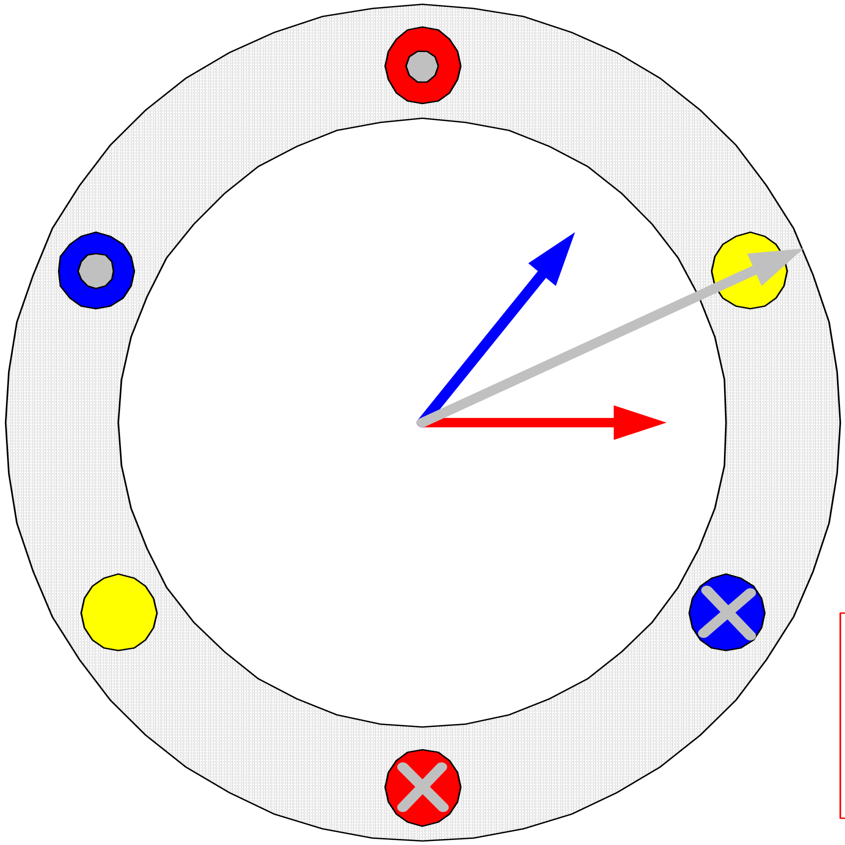


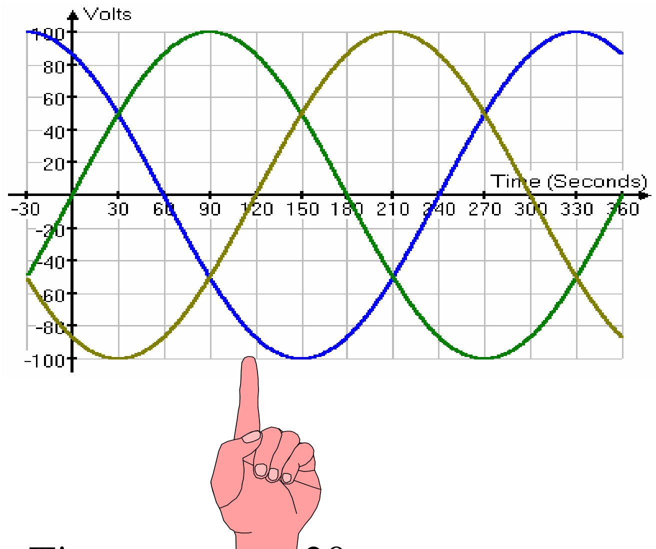


Time : t = 90 sec

Red phase = +100 V
Yellow Phase = - 50 V
Blue Phase = - 50 V

Magnetic Field produced in a 2 pole induction motor

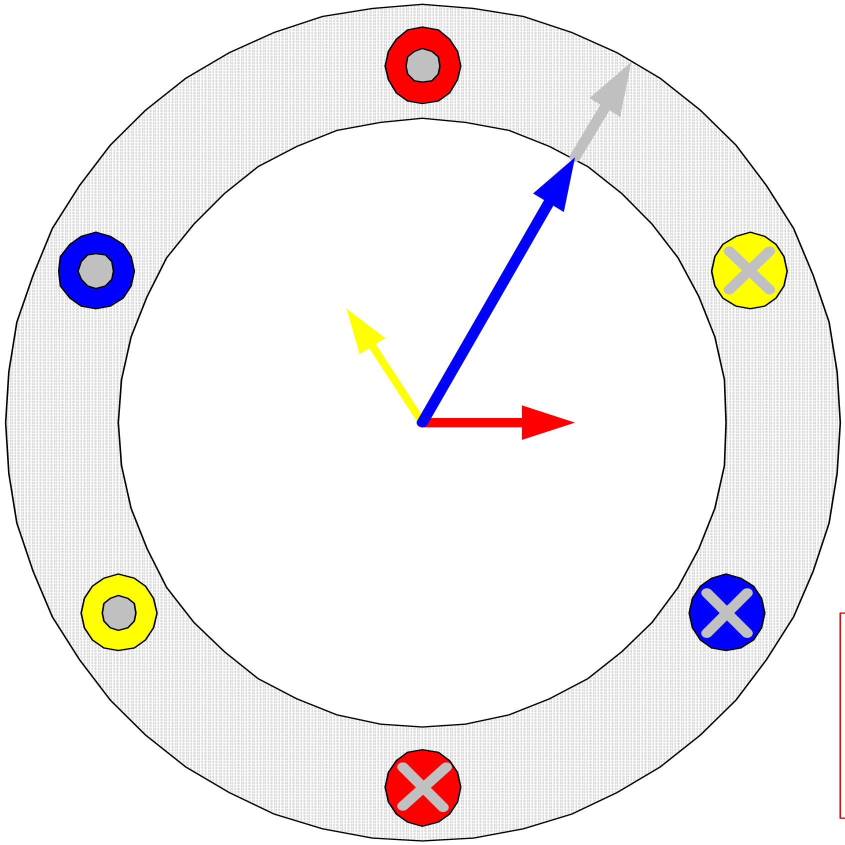


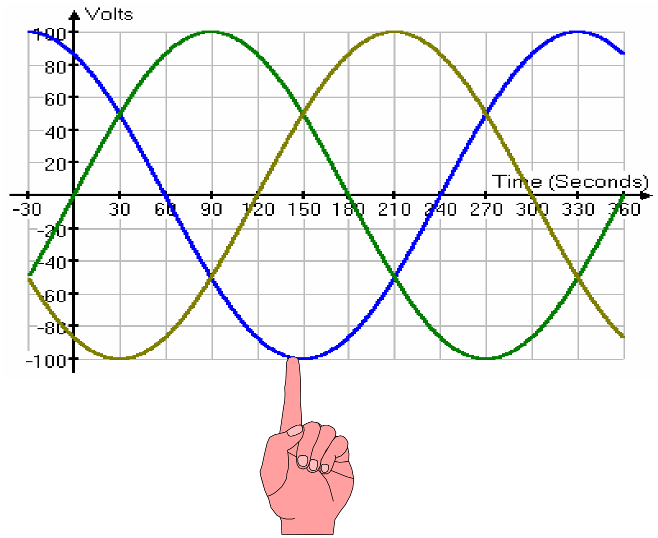


Time : t = 120 sec

Red phase = +86.6 V
Yellow Phase = 0 V
Blue Phase = -86.6 V

Magnetic Field produced in a 2 pole induction motor

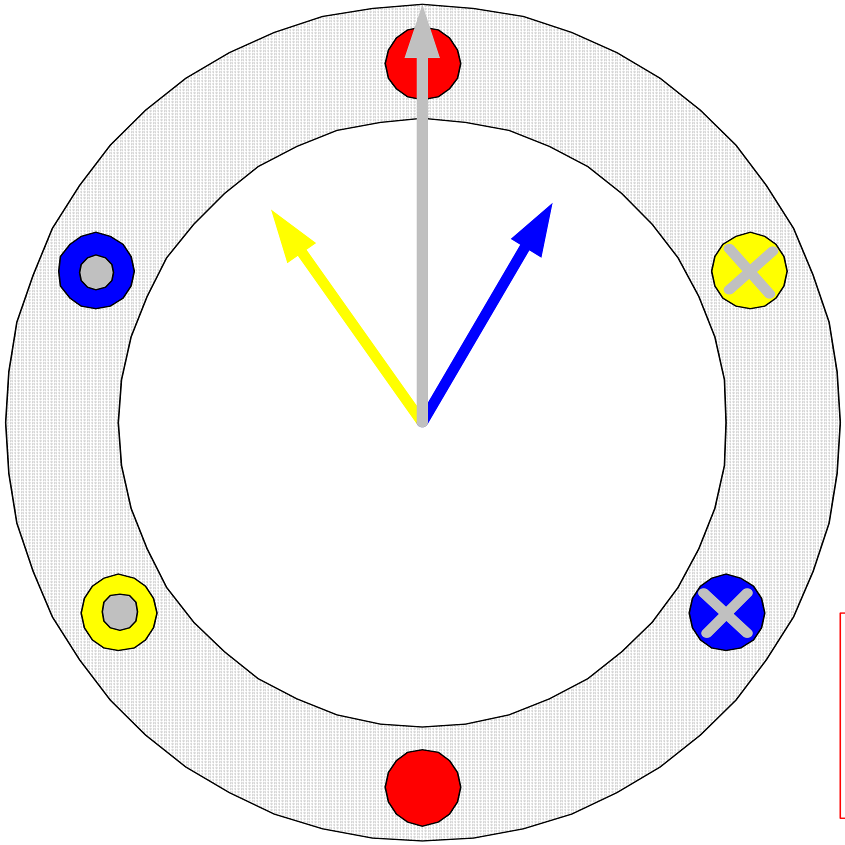


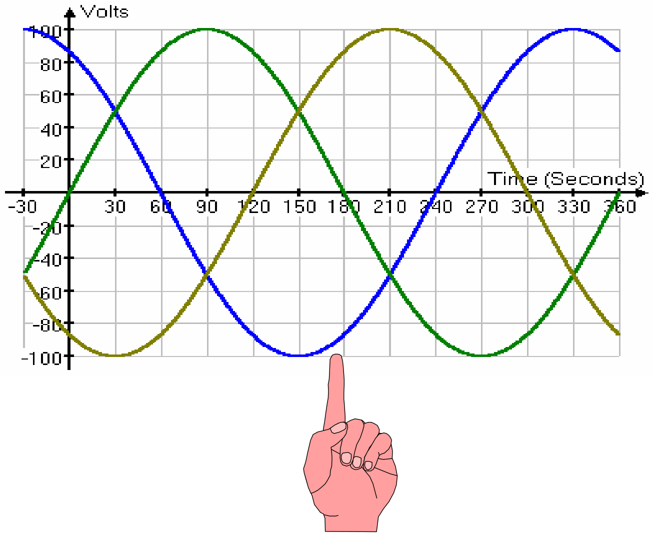


Time : t = 150 sec

Red phase = +50 V
Yellow Phase = +50 V
Blue Phase = -100 V

Magnetic Field produced in a 2 pole induction motor

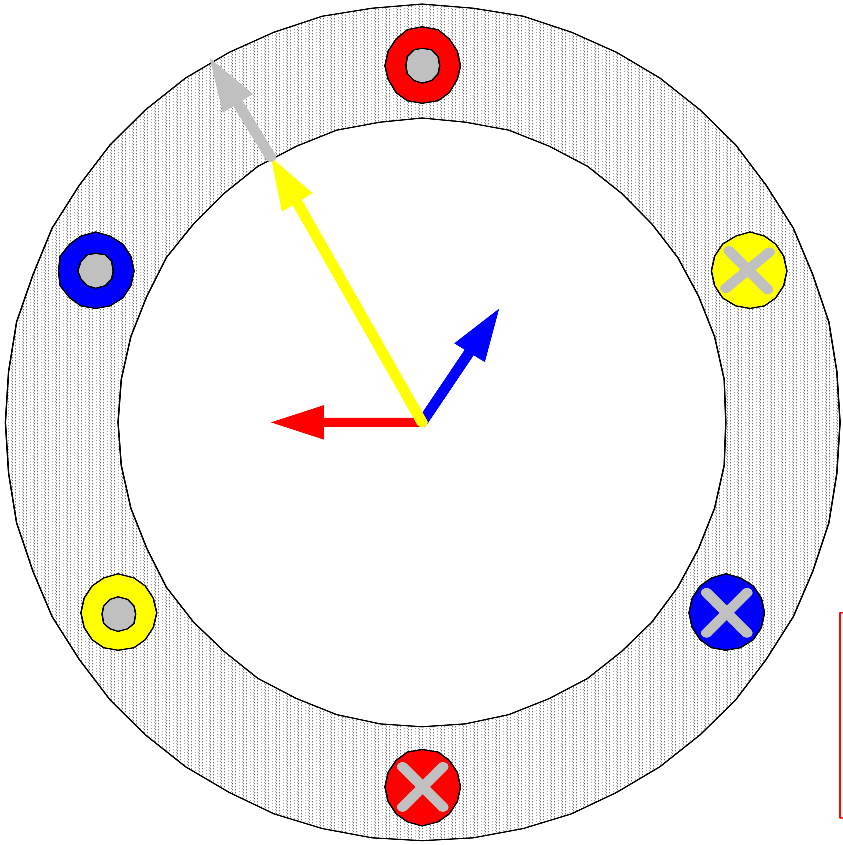


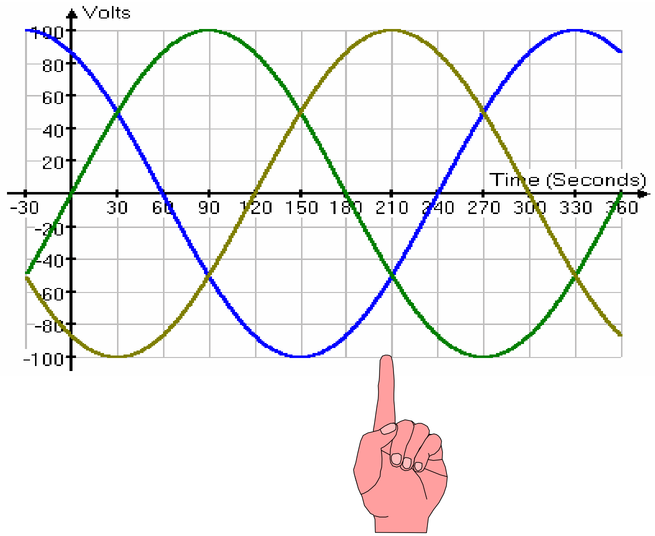


Time : t = 180 sec

Red phase = 0 V
Yellow Phase = +86.6 V
Blue Phase = -86.6 V

Magnetic Field produced in a 2 pole induction motor

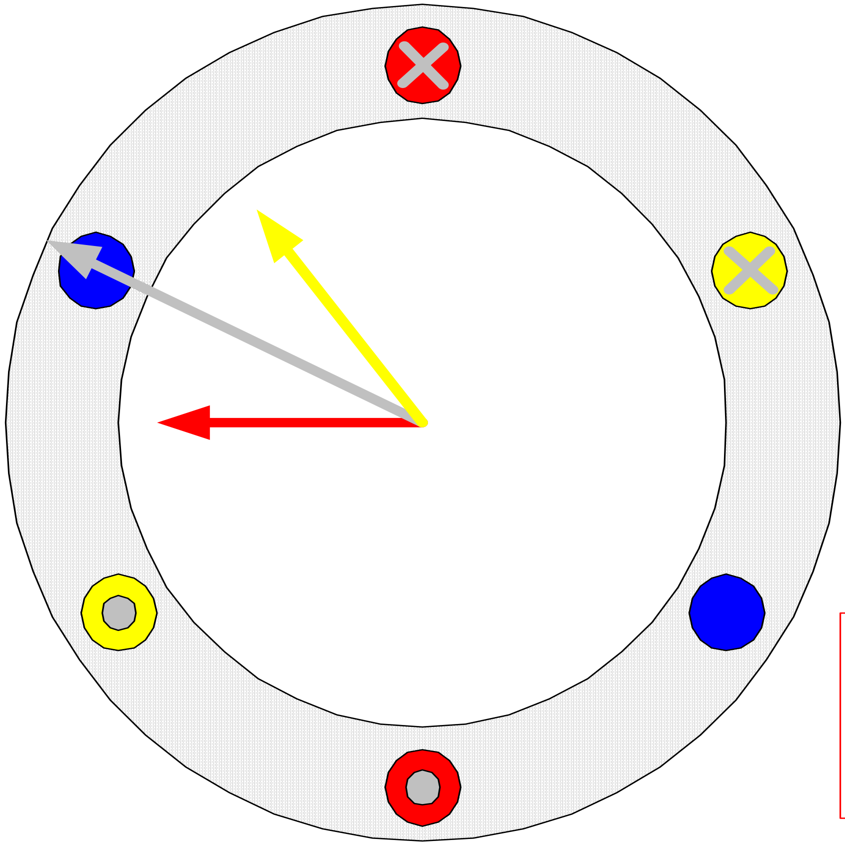


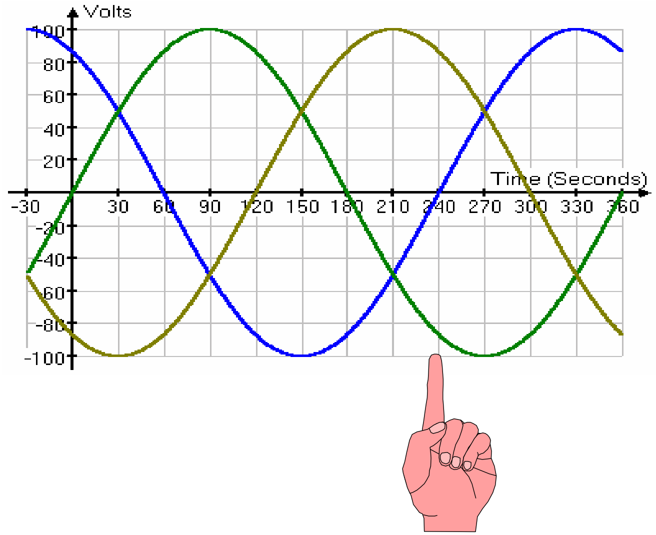


Time : t = 210 sec

Red phase = -50 V
Yellow Phase = +100 V
Blue Phase = -50 V

Magnetic Field produced in a 2 pole induction motor

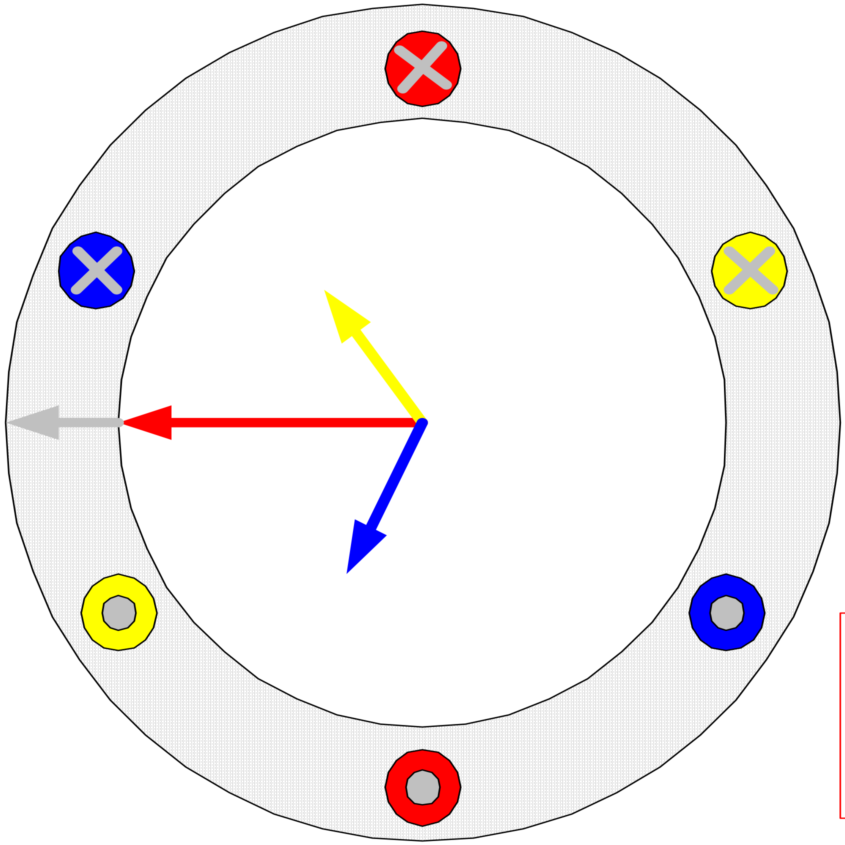


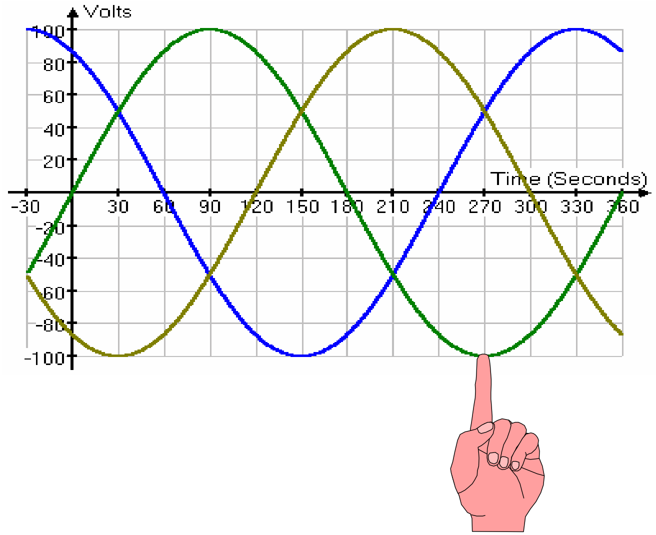


Time : t = 240 sec

Red phase = -86.6 V
Yellow Phase = +86.6 V
Blue Phase = 0 V

Magnetic Field produced in a 2 pole induction motor

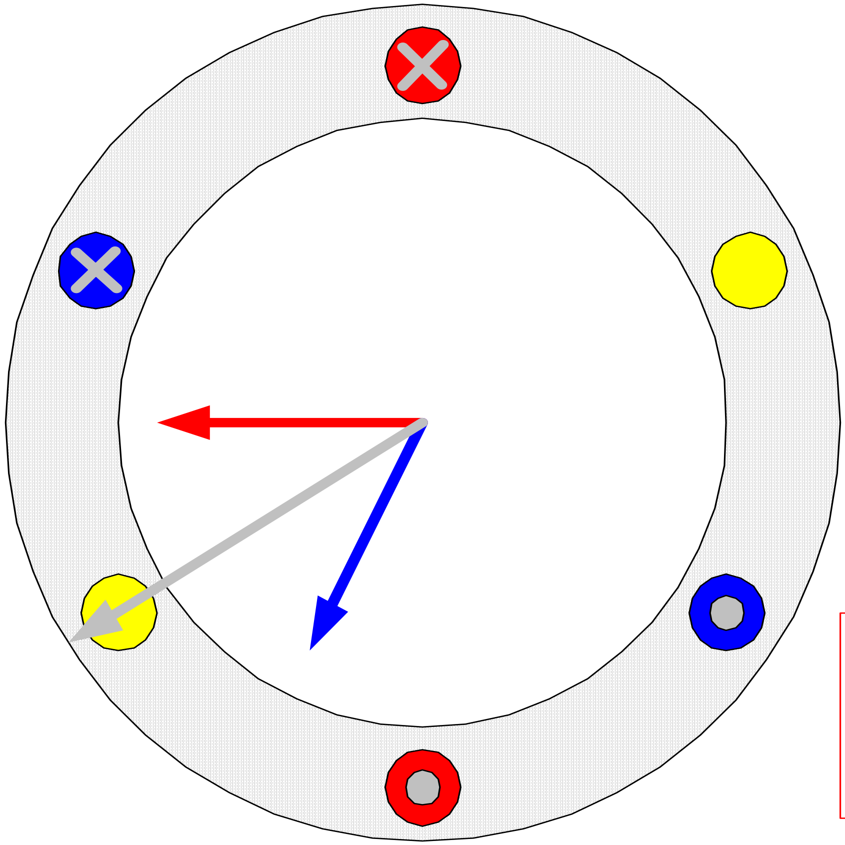


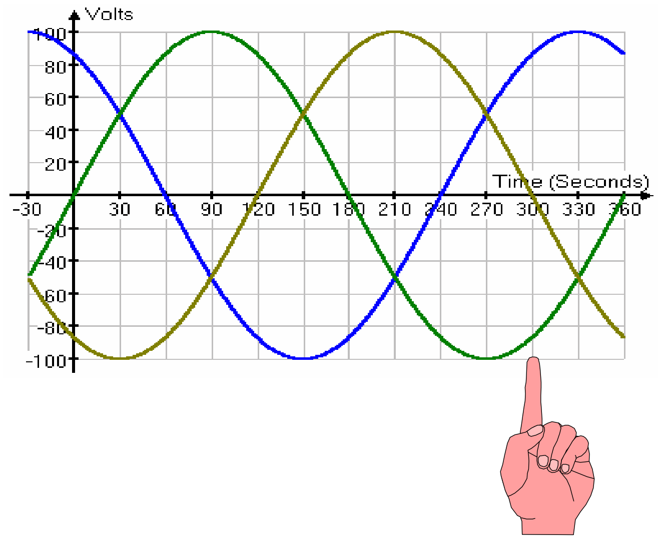


Time : t = 270 sec

Red phase = -100 V
Yellow Phase = +50 V
Blue Phase = +50 V

Magnetic Field produced in a 2 pole induction motor

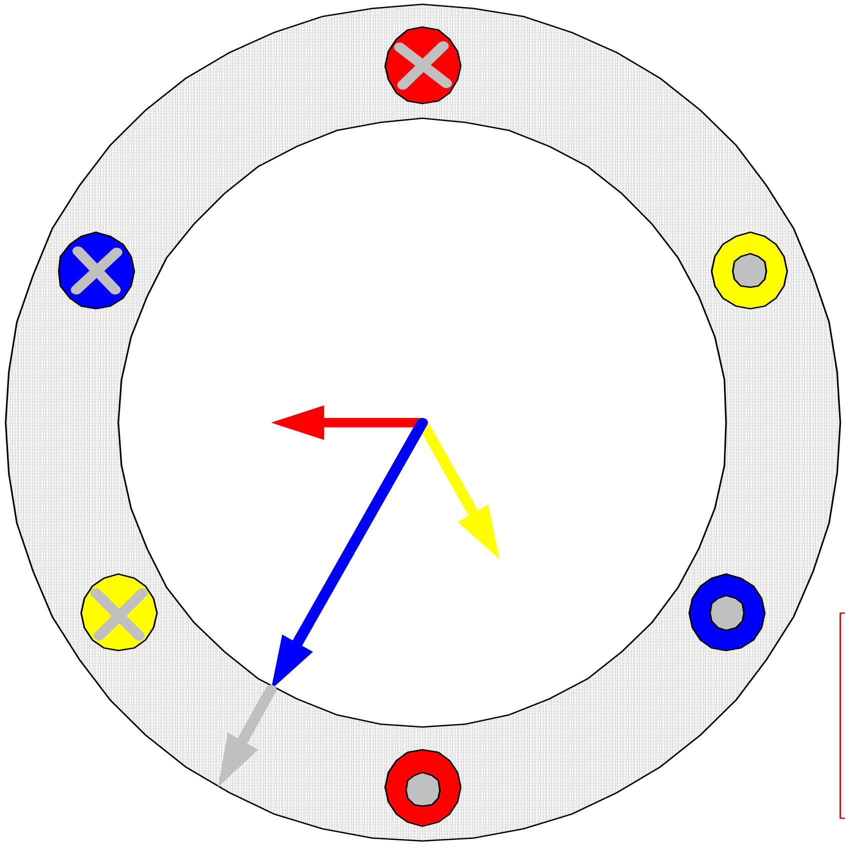


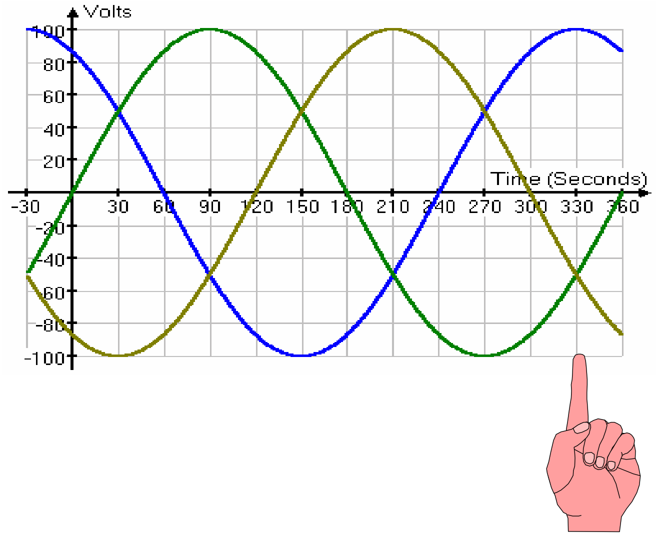


Time : t = 300 sec

Red phase = -86.6 V
Yellow Phase = 0 V
Blue Phase = +86.6 V

Magnetic Field produced in a 2 pole induction motor

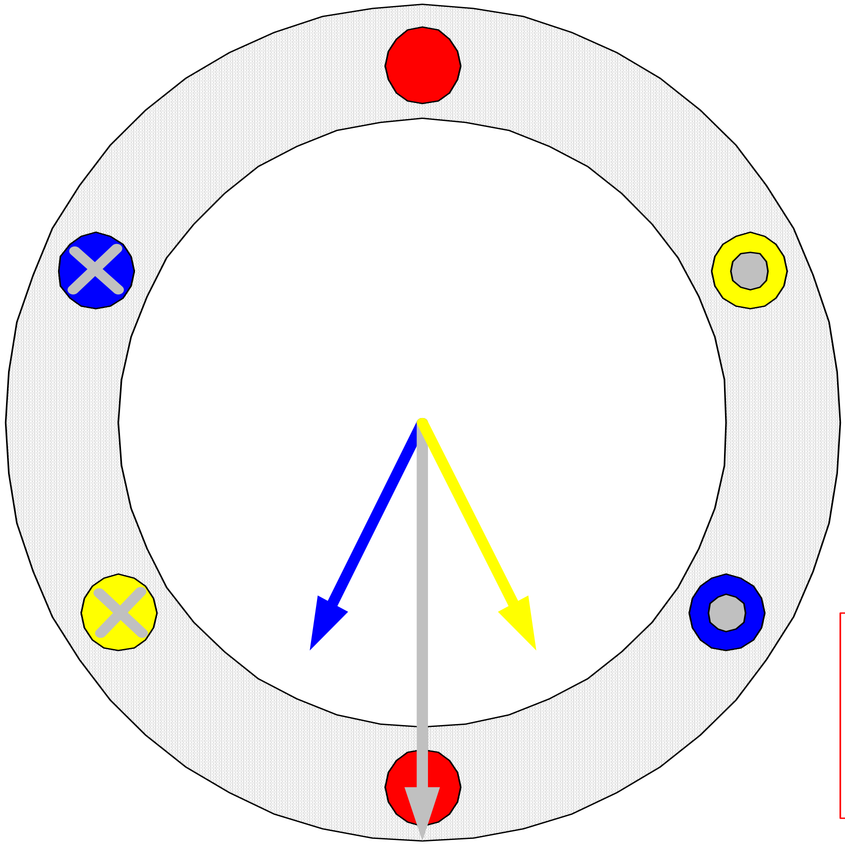


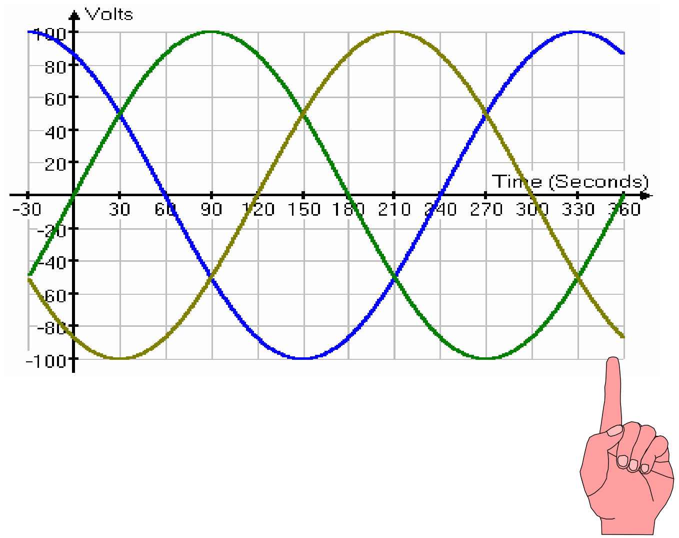


Time : t = 330 sec

Red phase = -50 V
Yellow Phase = -50 V
Blue Phase = +100 V

Magnetic Field produced in a 2 pole induction motor

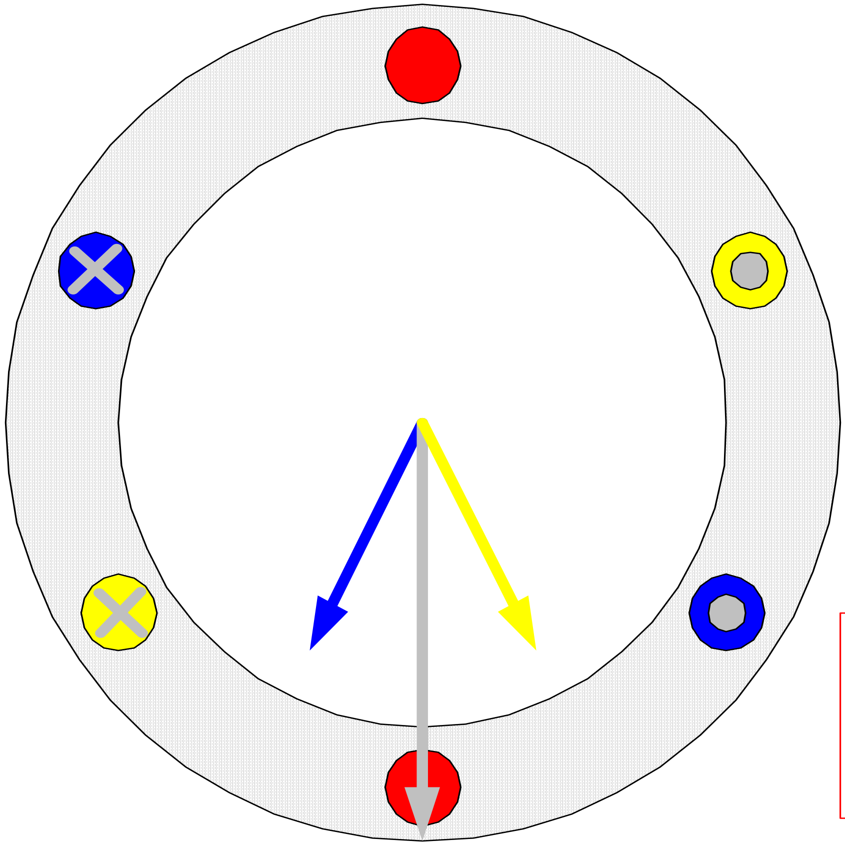


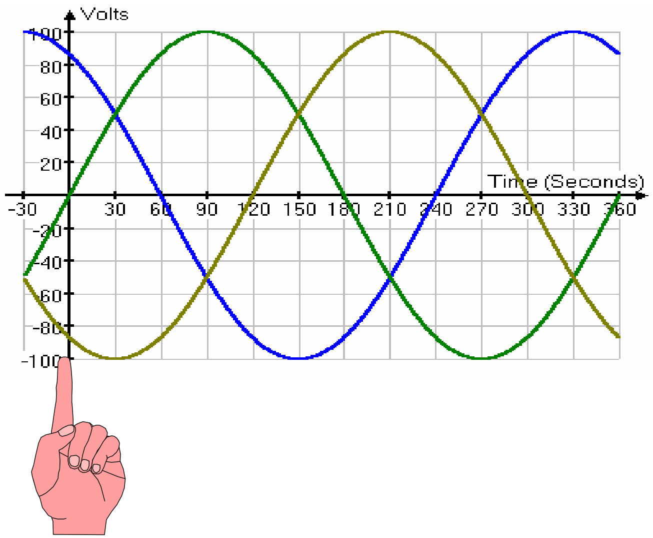


Time : t = 360 sec

Red phase = 0 V
Yellow Phase = -86.6 V
Blue Phase = +86.6 V

Magnetic Field produced in a 2 pole induction motor





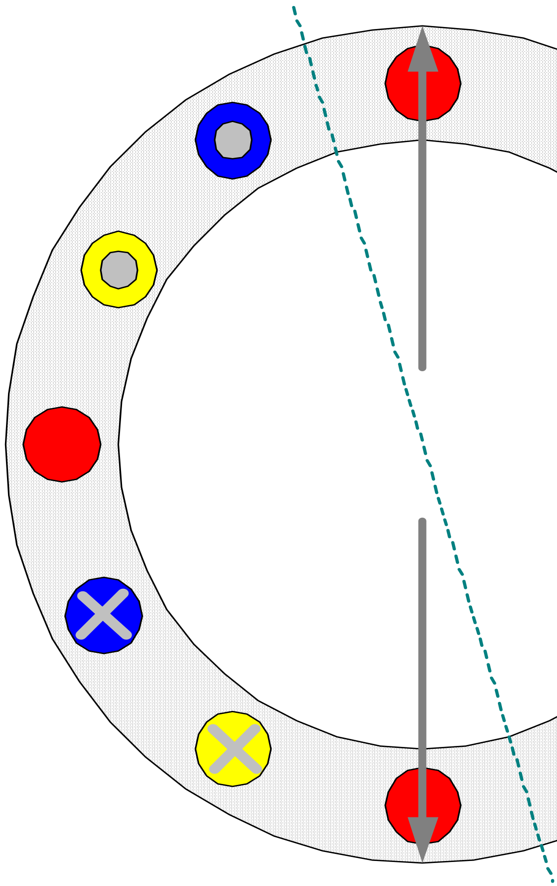
Time : t = 390

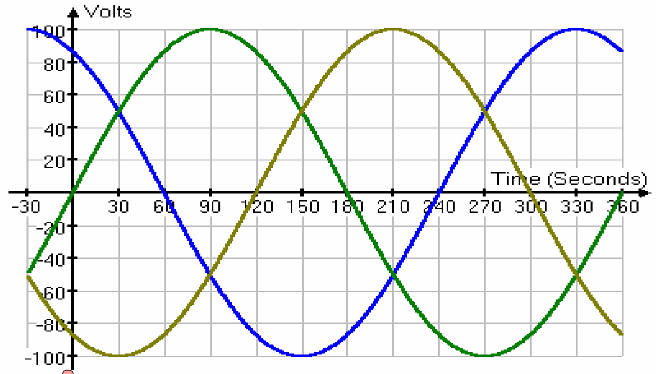
Red phase = 0 V
Yellow Phase = -86.6 V
Blue Phase = +86.6 V

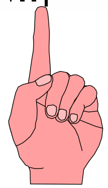
Speed of rotating field produced by 3-phase currents

- We understand that a complete sinusoidal cycle will cause the magnetic field produced to rotate a complete 360 degrees in a two pole 3-phase induction motor. Since there are 60 seconds in one minute, the speed of the rotating field is equal to $frequency \times 60 \sec onds$
- A basic 2-pole induction motor has a set of three winding groups installed in the stator (Red start & finish, Yellow start & finish and Blue start & finish). For an induction motor with 4 pairs of poles, there are two sets of three winding groups installed in the stator, as a result, the magnetic field only rotates 180 degrees for every complete cycle of the sinusoidal waveform. Therefore, for an induction motor with P pairs of poles, the speed of the rotating field is given by: $\dfrac{frequency \times 60 \sec onds}{pole - pairs}$

Magnetic Field produced by three phase currents in a four pole induction motor



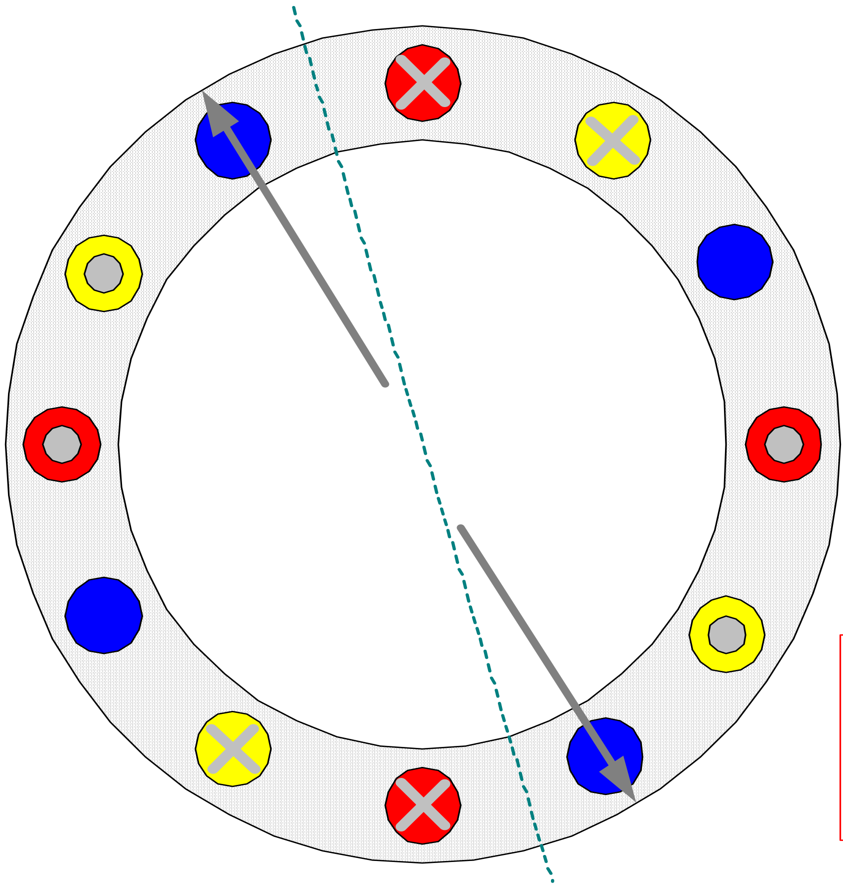


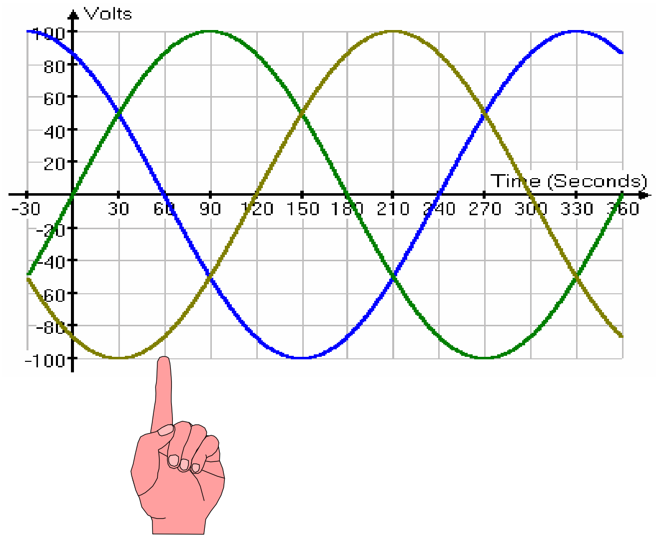


Time : t = 0

Red phase = 0 V
Yellow Phase = -86.6 V
Blue Phase = +86.6 V

Magnetic Field produced by three phase currents in a four pole induction motor

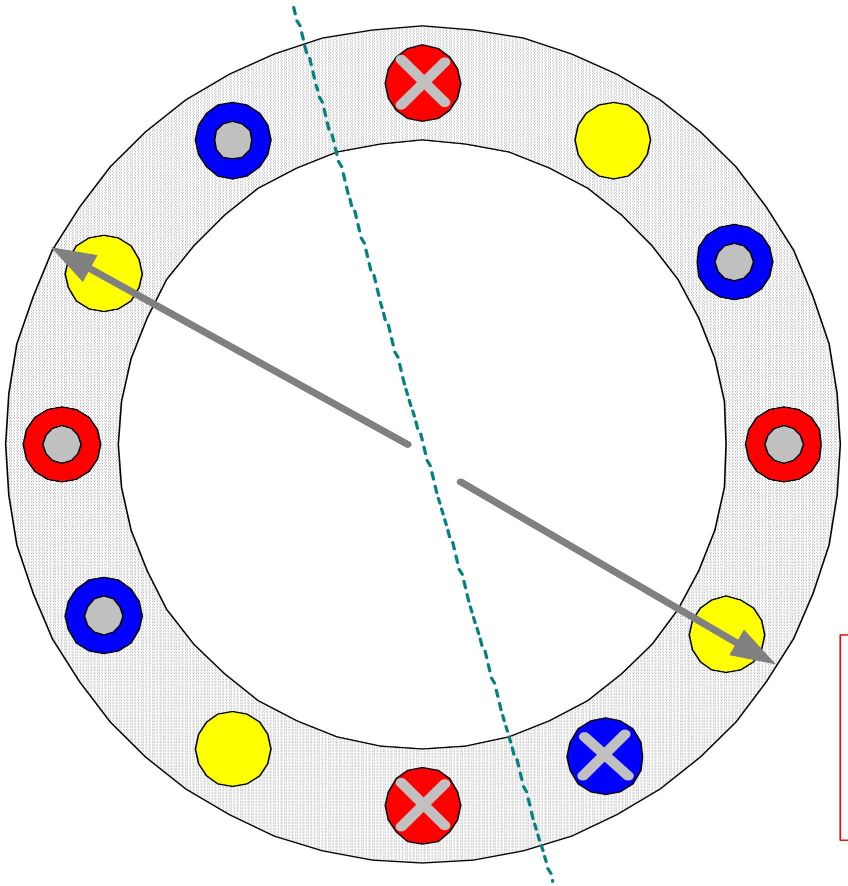


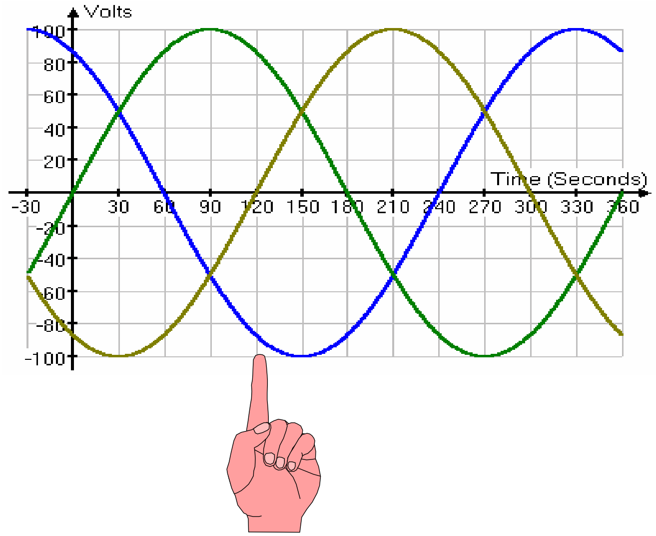


Time : t = 60

Red phase = +86.6 V
Yellow Phase = -86.6 V
Blue Phase = 0 V

Magnetic Field produced by three phase currents in a four pole induction motor

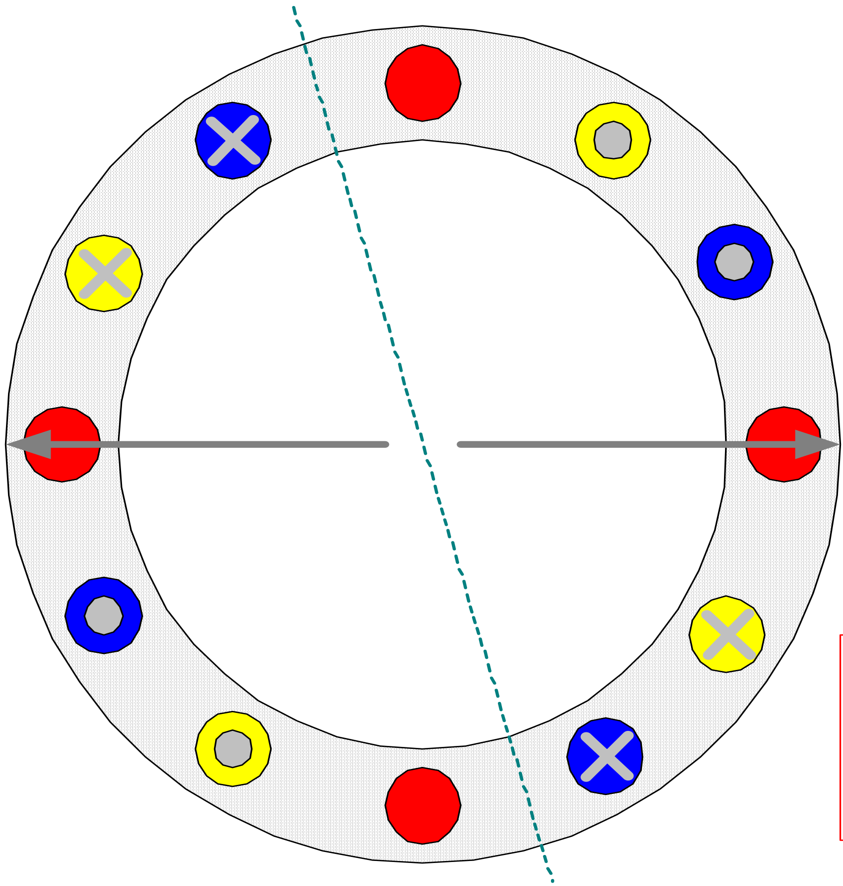


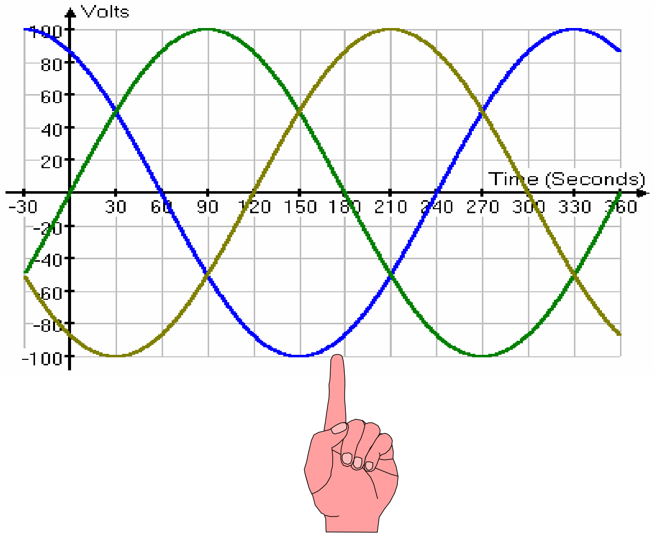


Time : t = 120

Red phase = +86.6 V
Yellow Phase = 0 V
Blue Phase = -86.6 V

Magnetic Field produced by three phase currents in a four pole induction motor

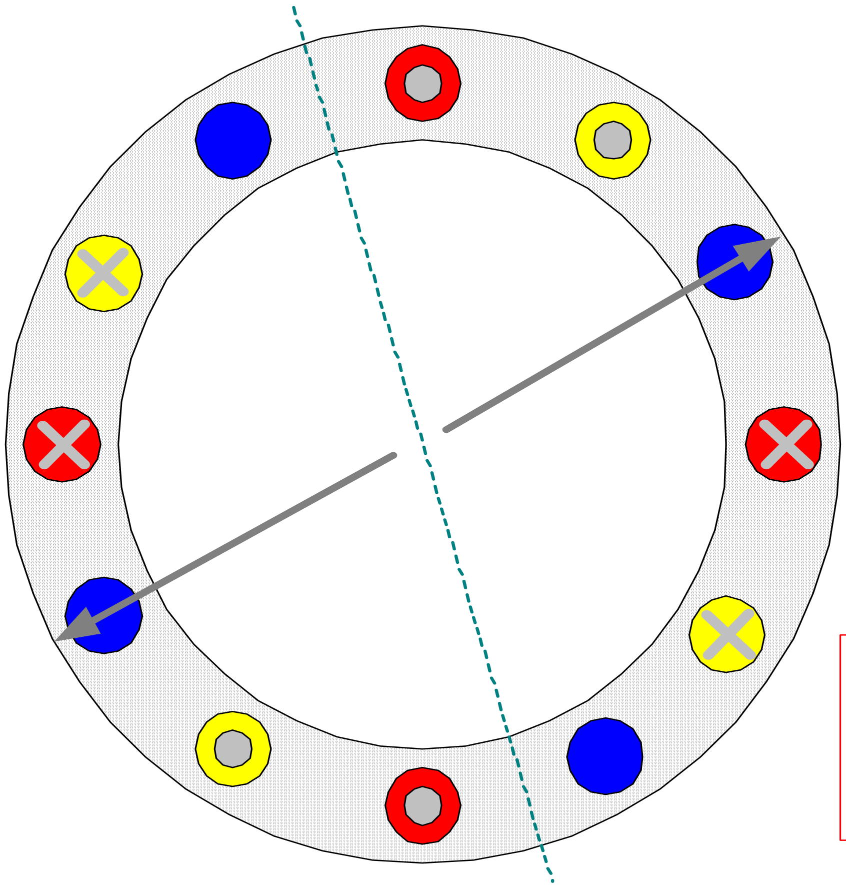


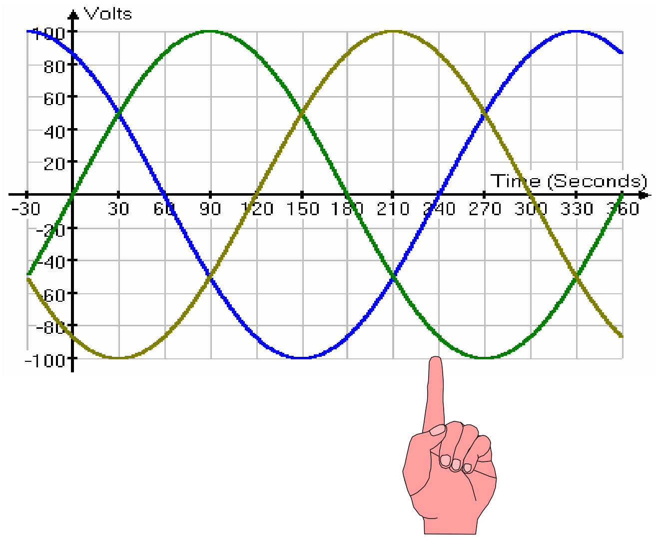


Time : t = 180

Red phase = 0 V
Yellow Phase = +86.6 V
Blue Phase = -86.6 V

Magnetic Field produced by three phase currents in a four pole induction motor

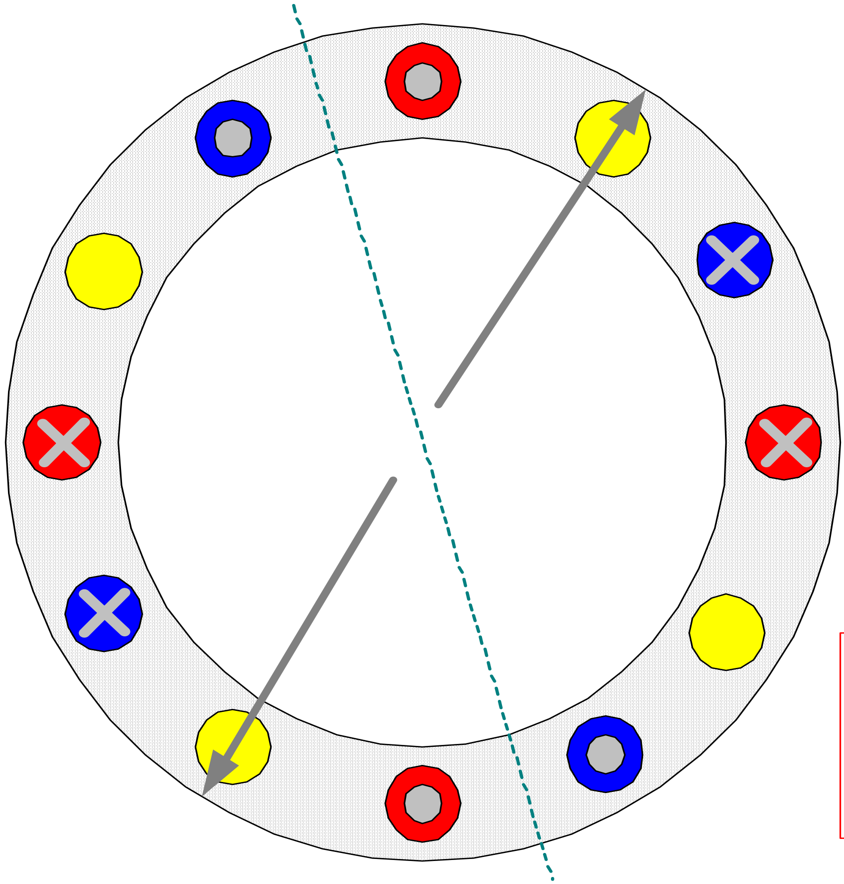


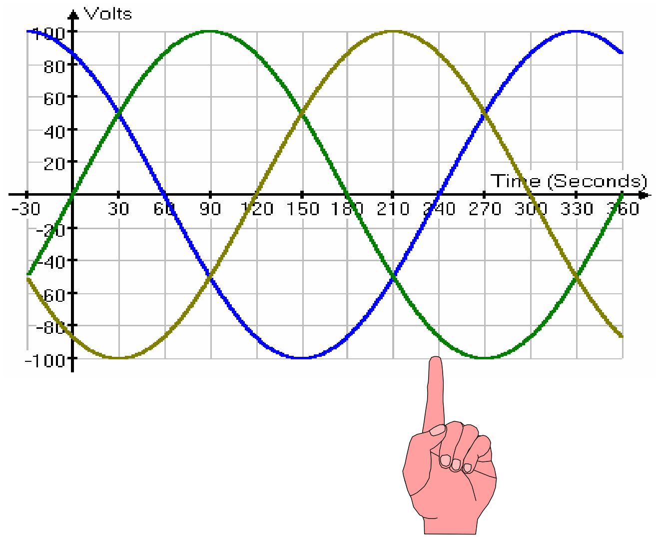


Time : t = 240

Red phase = -86.6 V
Yellow Phase = +86.6 V
Blue Phase = 0 V

Magnetic Field produced by three phase currents in a four pole induction motor

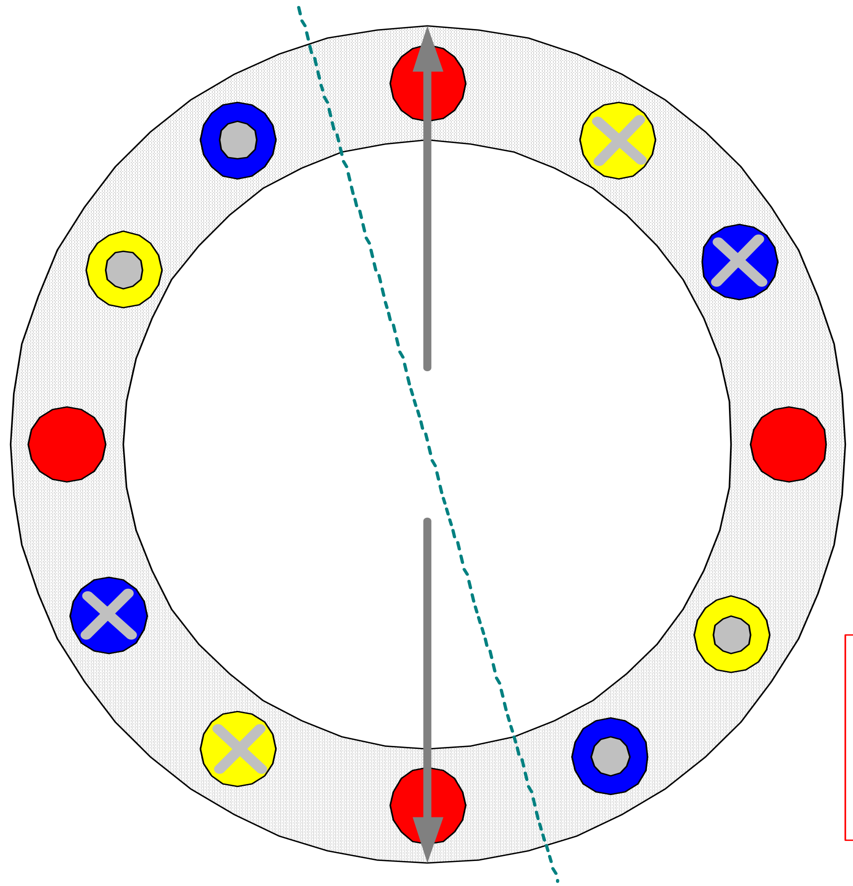


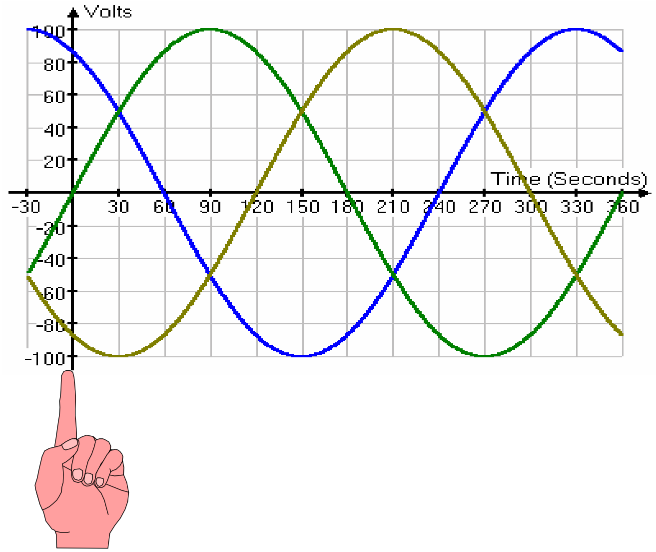


Time : t = 300

Red phase = -86.6 V
Yellow Phase = 0 V
Blue Phase = +86.6 V

Magnetic Field produced by three phase currents in a four pole induction motor





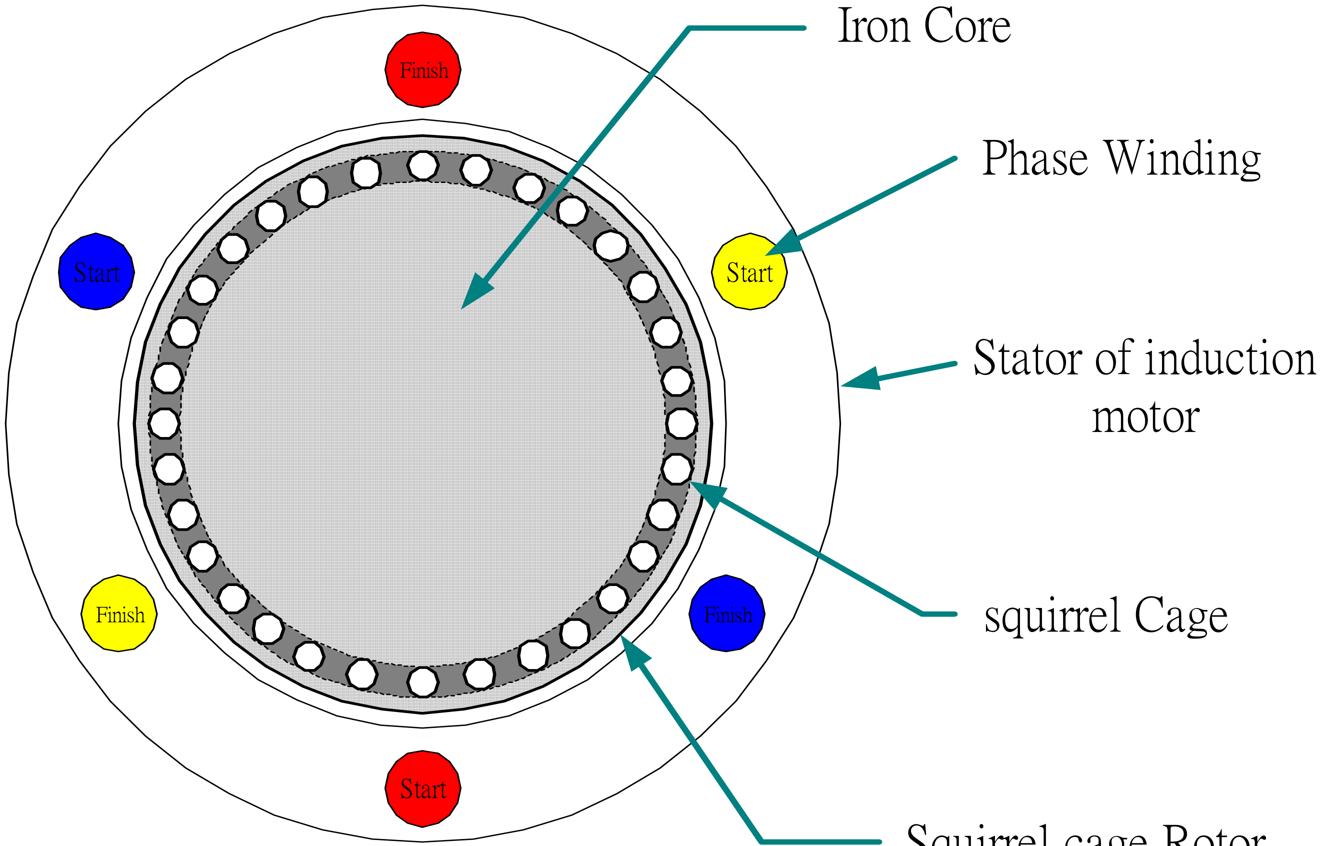
Time : t = 360

Red phase = 0 V
Yellow Phase = -86.6 V
Blue Phase = +86.6 V

Production of rotational torque in three phase induction motor

- A rotating magnetic field is produced when a three phase voltage is fed to the stator of a three phase induction motor
- This rotating field will traverse the aluminium conductors of the squirrel cage rotor
- According to Fleming's Right hand rule for generator, e.m.f. is induced in the aluminium bars of the squirrel cage
- According to Fleming's Left hand rule for motor, a torque is produced which will drive the rotor rotating in the same direction as the magnetic field

Cross- Section of a three phase squirrel cage inductor motor



The rotor of a squirrel cage induction motor

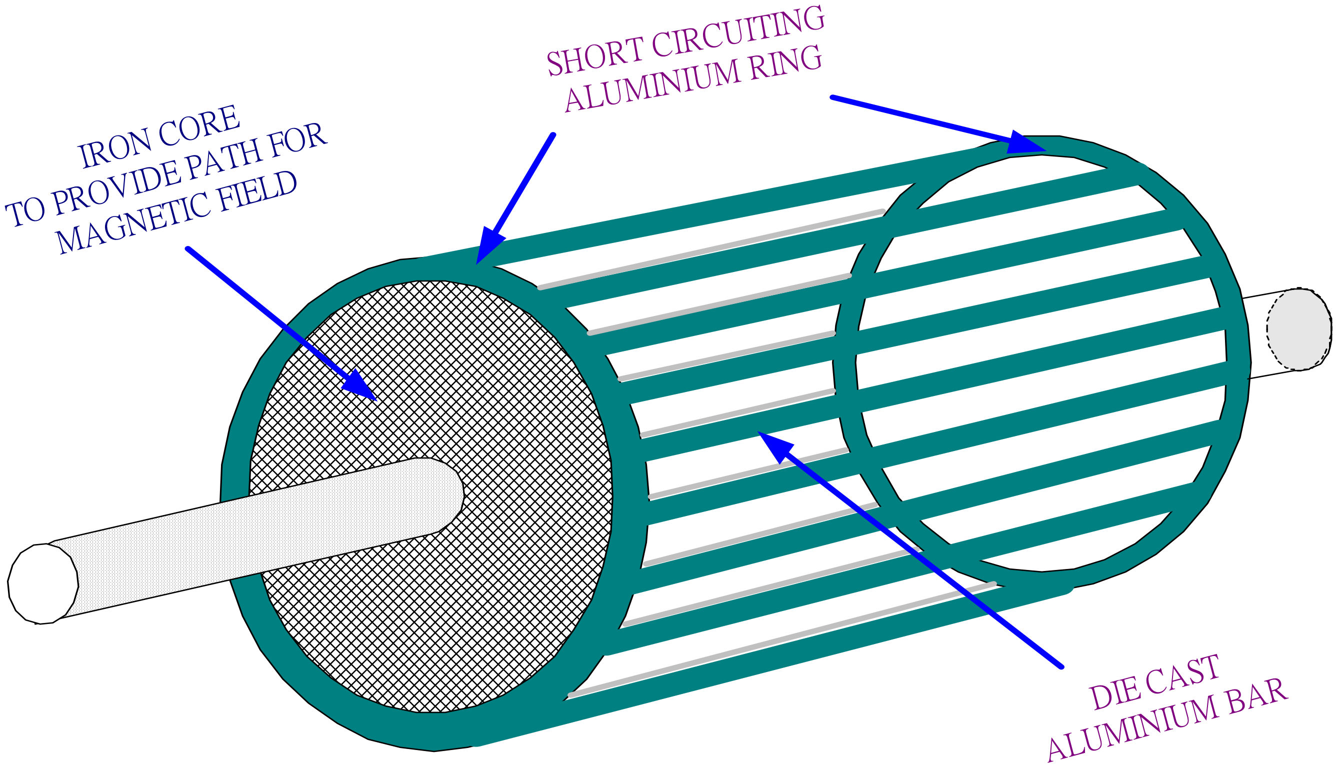
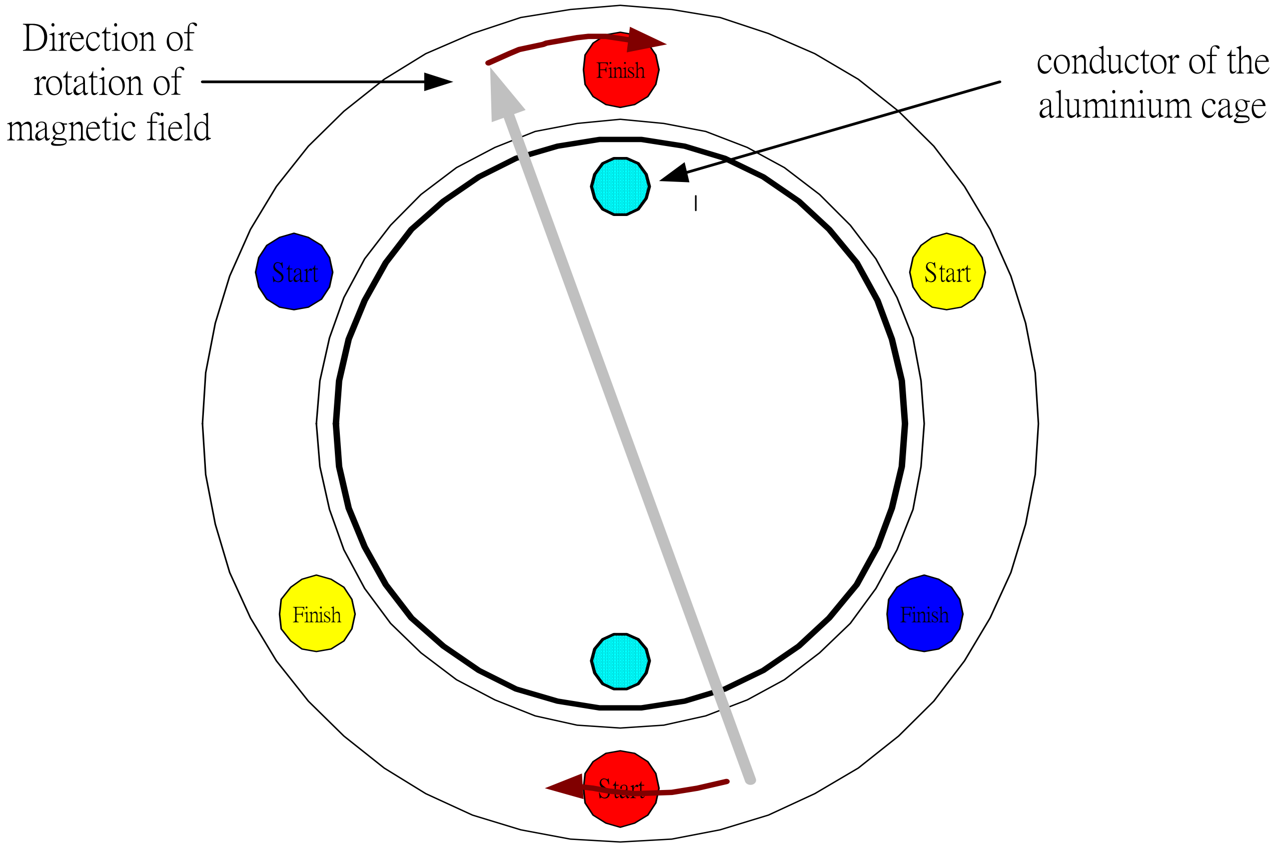
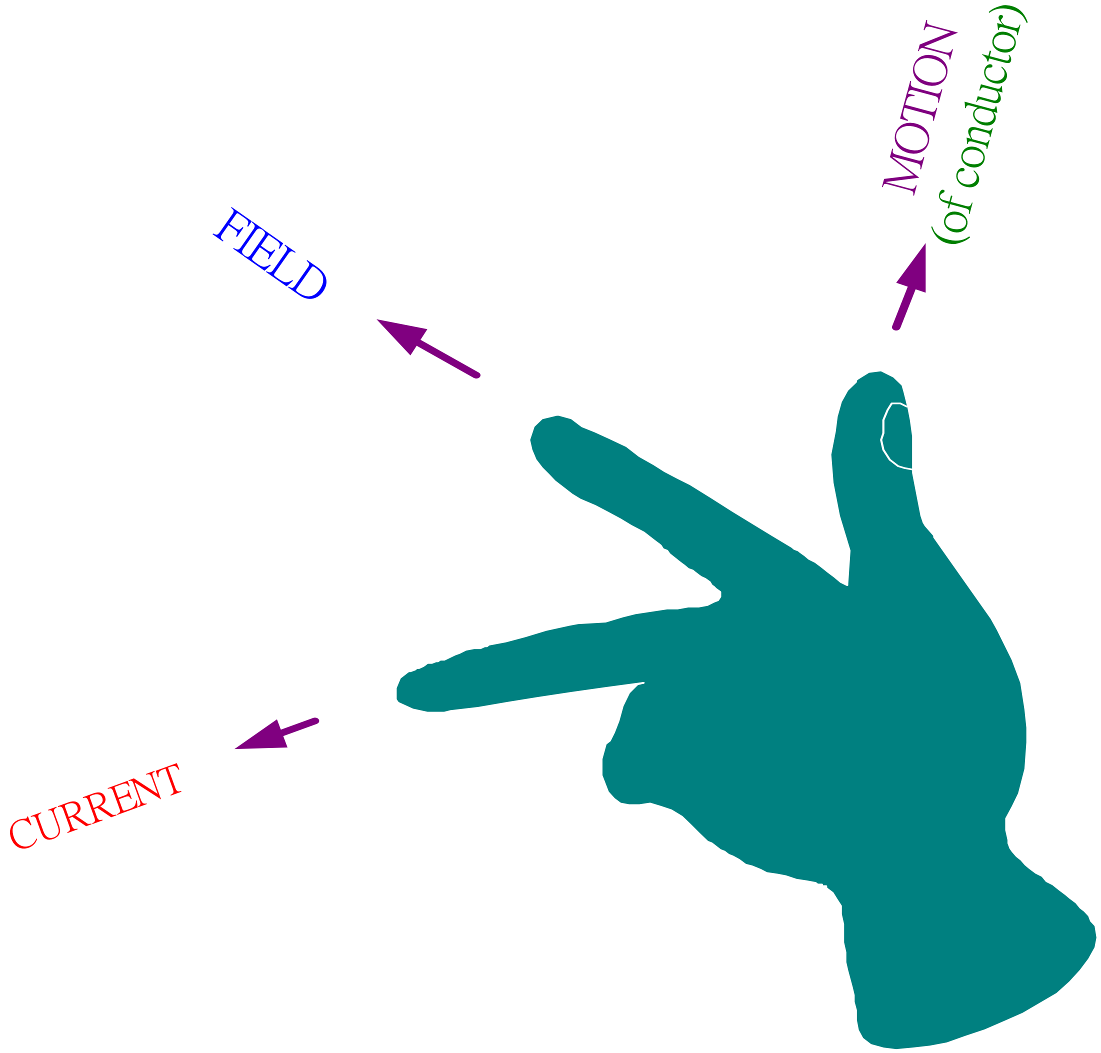


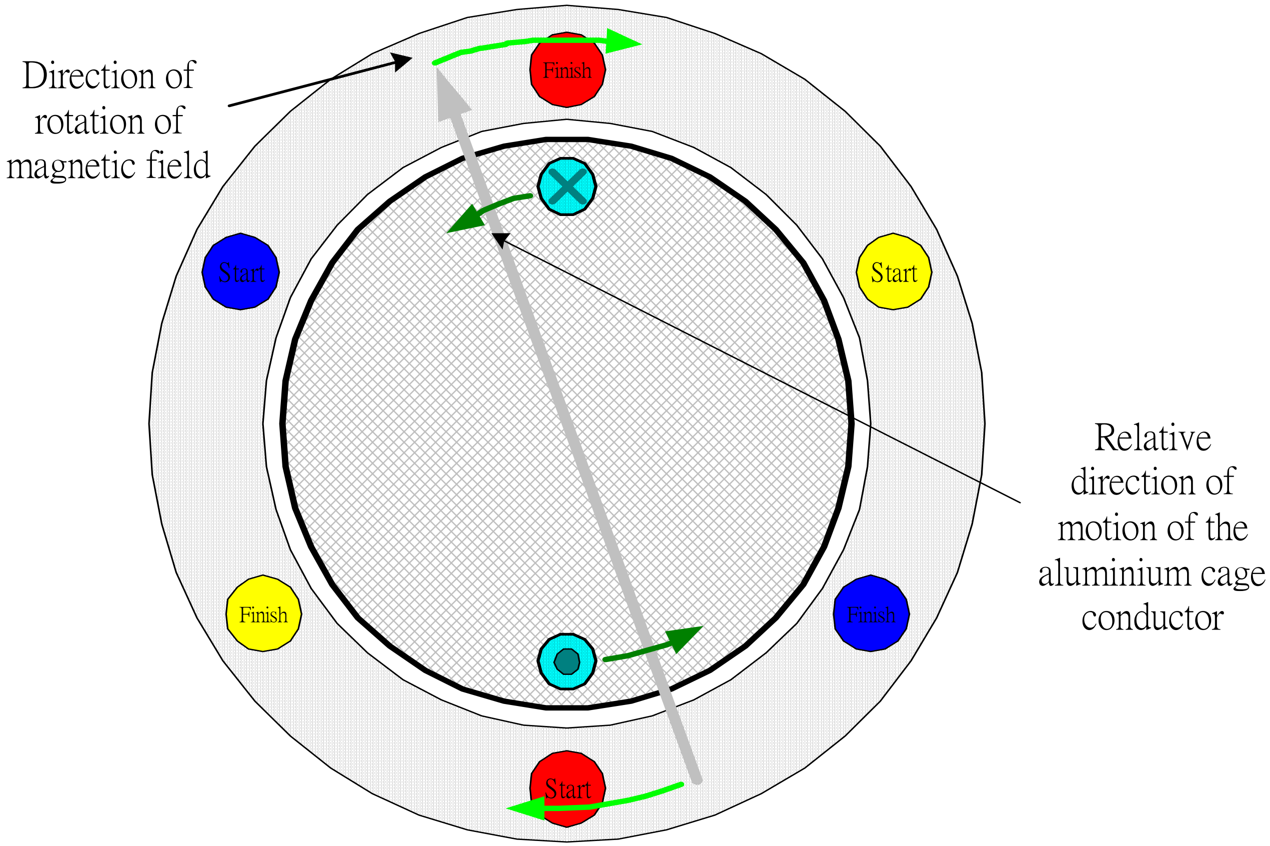
Diagram showing the squirrel cage conductor 'cut' by the rotating magnetic field



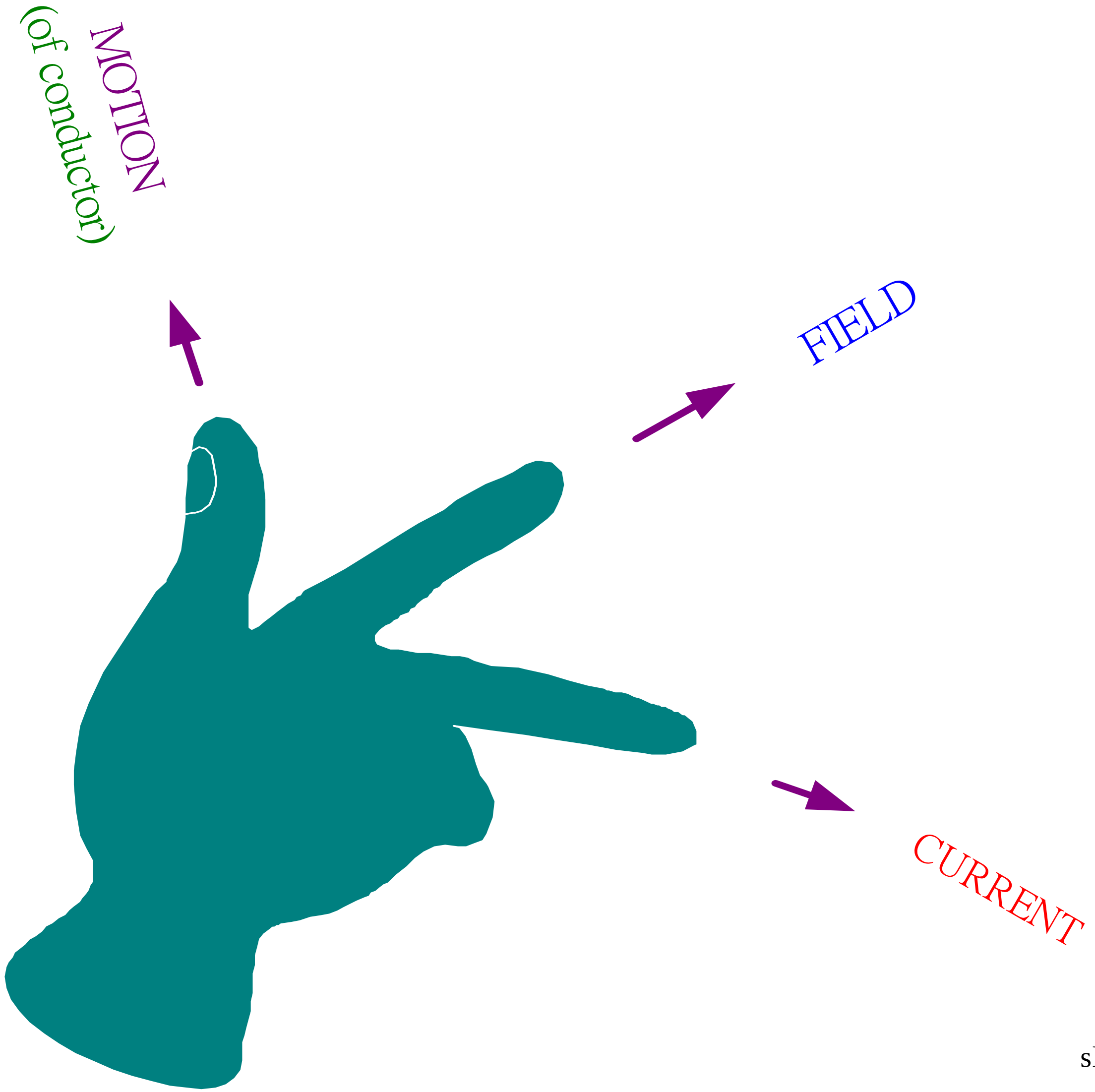
Fleming’s Right Hand Rule for generator



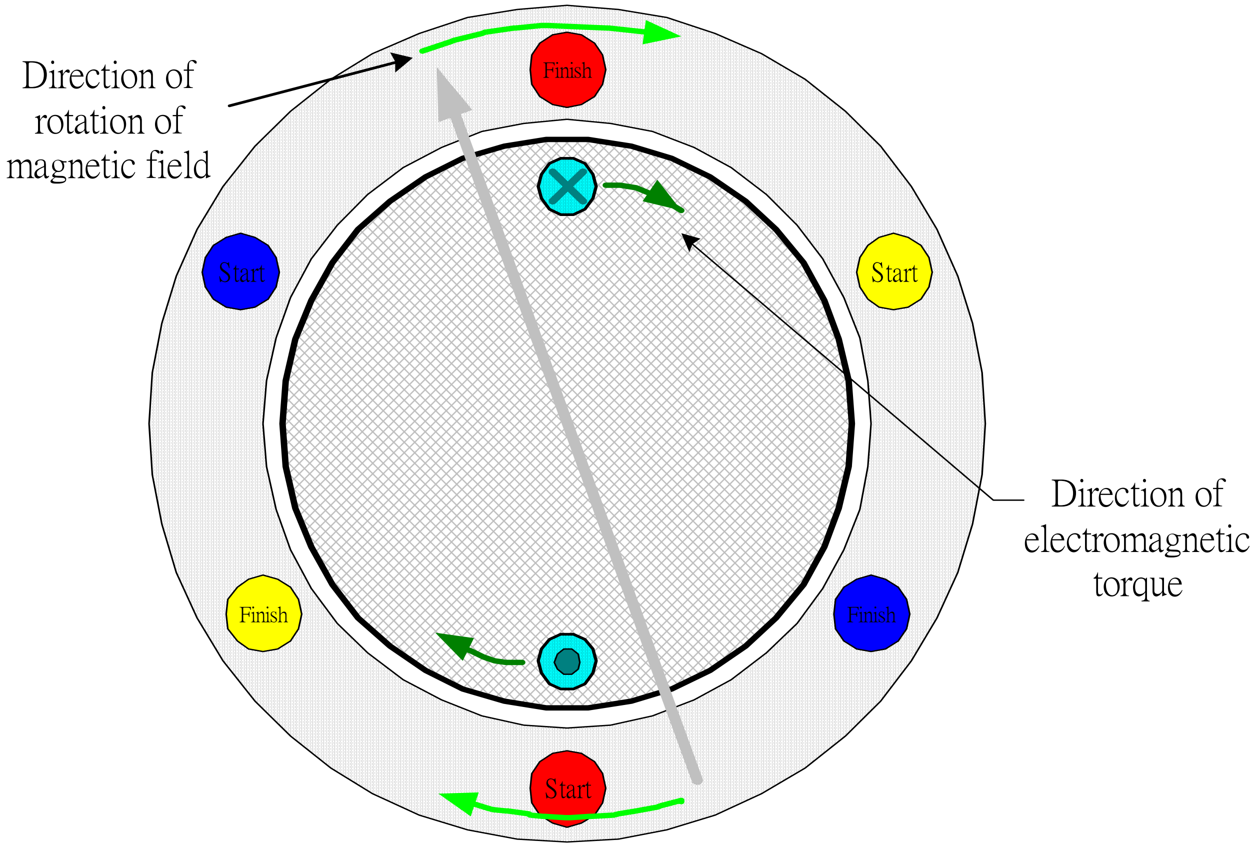
Apply Fleming's Right Hand rule to determine the direction of induced current on the rotor cage



Fleming's Left Hand Rule for motor



Apply Fleming's Right Hand rule to determine the direction of induced current on the rotor cage



Definition of Slip for an induction motor

- The magnetic field of an induction motor rotates at a synchronous speed Ns which is equal to $\frac{frequency \times 60 \sec onds}{pole - pairs}$
- This magnetic field will induce current in the rotor circuit, causing the rotor to run in the same direction as the field
- However, the speed of the rotor Nr is always slower than the speed of the field. Since if the speed of the rotor is equal to that of the field there will be no induced e.m.f. and there will be no driving torque to keep the rotor running
- Percentage slip is defined as $\frac{Ns - Nr}{Ns} \times 100\%$

Example of Slip calculation

Example

- Calculate the slip of an 8-pole, 3-phase induction motor running at 846 rev / min. The frequency of the three phase supply is 60 Hertz

Now Synchronous speed Ns = $\frac{frequency \times 60 \sec onds}{pole - pairs}$

$$= \frac{60 \times 60}{4} = 900$$

Rotor speed Nr = 846

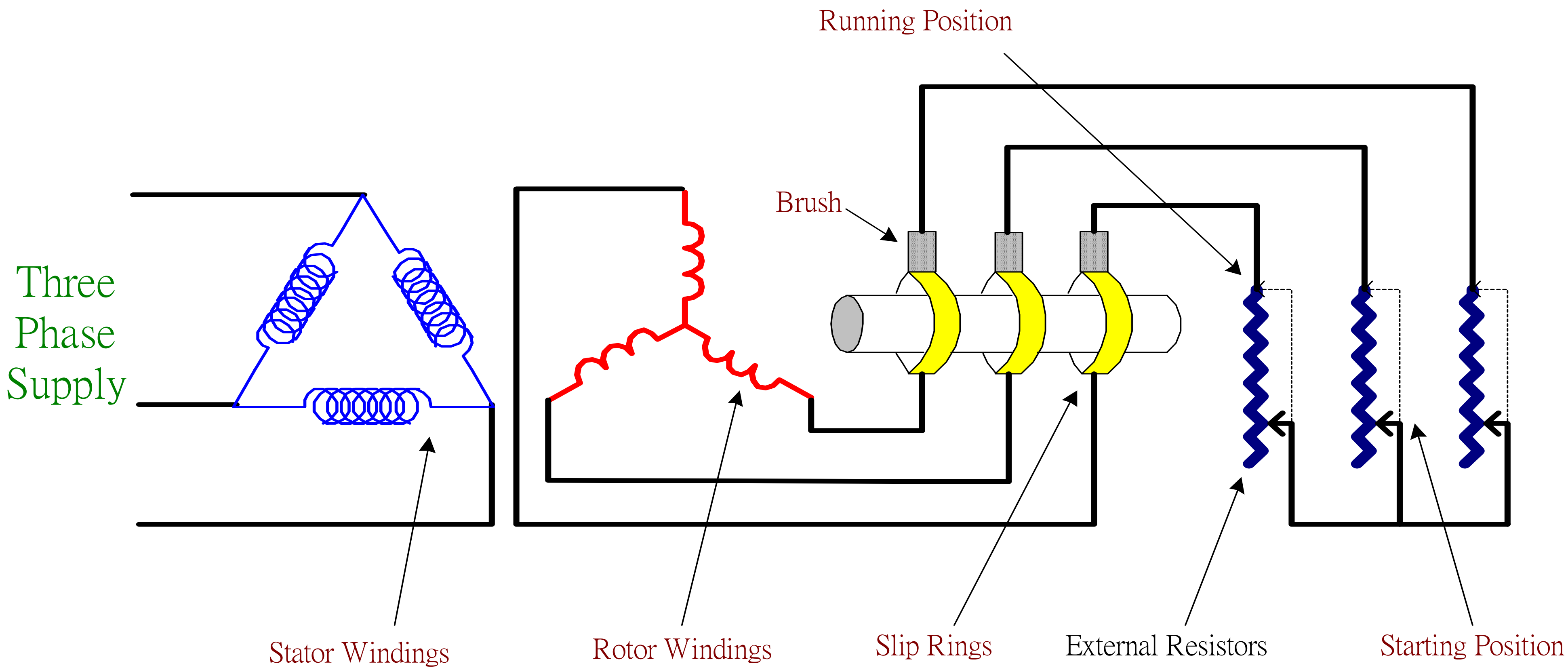
Therefore Slip S = $\frac{Ns - Nr}{Ns} \times 100\%$

$$= \frac{900 - 846}{900} \times 100\% = \frac{54}{900} \times 100\% = 6\%$$

Three phase Slip ring induction motor

- The stator construction of a slip ring induction motor is similar to that of a squirrel cage induction motor. A cage rotor is made of die cast aluminium. A slip ring rotor is wound with star connected three phase windings, the winding ends are brought out and connected to the exterior through three slip rings.
- The cage rotor is easy to manufacture, hence it is cheap and robust. However, we cannot do anything with the rotor circuit, in other words we cannot control the speed or starting torque of a cage rotor.
- A slip ring rotor is expensive to manufacture and it is vunerable to overheat, but we can connect suitable external resistance through the slip rings to the rotor circuit. As a result, we can control the starting torque and running characteristic of the slip ring motor.

Schematic Diagram of a wound rotor slip ring motor



Frequency of induced rotor current

- E.m.f. and current is induced in the rotor circuit when the rotor conductor is cut by the magnetic field at a rate which is equal to the difference in speed of the rotating magnetic field and the rotor
- When the rotor is not rotating (slip S = 1), we can treat the induction motor as a 3-phase transformer, the frequency of the induced e.m.f. = supply frequency f
- When the slip of the motor is S, the rotor is running at a speed $N_r = N_s(1 - S)$, e.m.f. is induced in the rotor conductor at a rate which is equal to the difference between the rotating field and the rotor speed.
  This rate is equal to $N_s - N_r = N_s - N_s(1 - S) = S\,N_s = \dfrac{S \times 60 \times frequency}{pole - pair}\ rev/minute$
  In other words, the frequency of the induced e.m.f. in the rotor circuit is equal to $\dfrac{S \times 60 \times frequency}{pole - pair} \times \dfrac{pole - pair}{60}\ Hertz = S \times frequency$

Impedance of rotor circuit

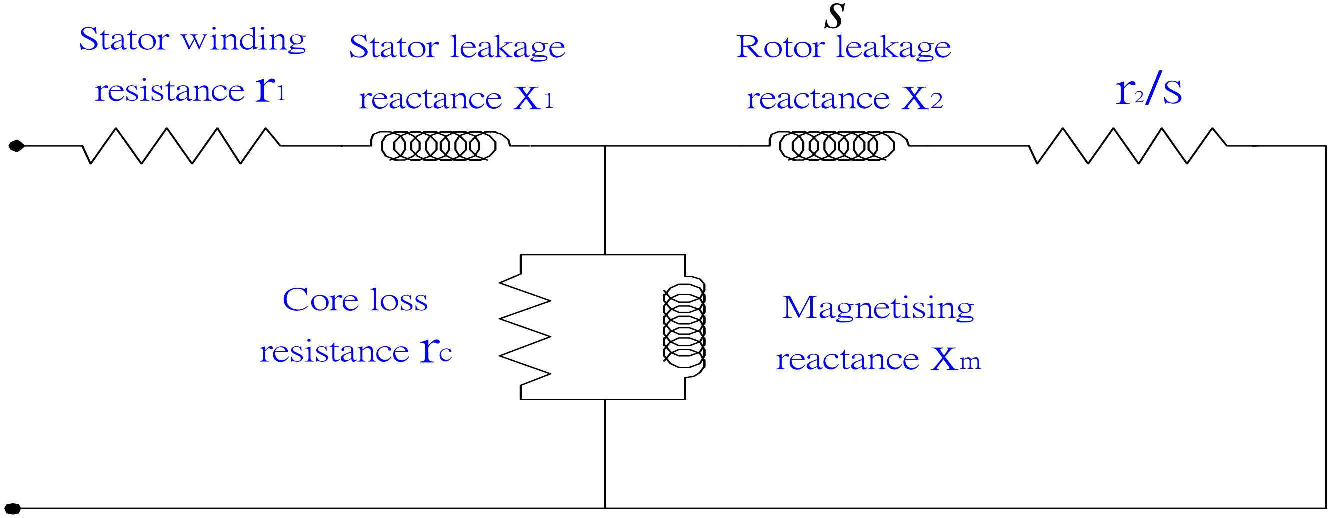
- The rotor circuit consists of conductor resistance and reluctance.
- The resistance of the rotor circuit is independent upon the frequency of the rotor current, it always has a value equal to r_2 (the suffix 2 represent secondary)
- The reactance (= $2 \times \pi \times f$) of the rotor circuit, however depends on the frequency of the rotor induced current:
  - When the rotor is at stationary, the rotor current is at supply frequency, the rotor reactance is equal to $2 \times \pi \times s \times f = s \times x_2$
  - When the rotor is running at slip S, the frequency of the rotor current is equal to $s \times f$, the rotor reactance is equal to $= 2 \times \pi \times f = x_2$

Induced E.m.f. and current in the rotor circuit

- When the rotor is at stationary, the E.m.f. induced in the rotor circuit has a value which is equal to $= 4.44 \times f \times N_2 \times \phi = E_2$ where N_2 is the effective number of turns of the rotor circuit.
- When the rotor is running at slip S, the E.m.f. induced in the rotor circuit has a value which is equal to $= 4.44 \times S \times f \times N_2 \times \phi = S \times E_2$
- The rotor current I_2 is equal to the rotor induced E.m.f. divided by the rotor circuit impendance:
  - When the rotor is at stationary, $I_2 = \dfrac{E_2}{\sqrt{r_2^2 + x_2^2}}$
  - When the rotor is running at slip S, $I_r = \dfrac{E_r}{Z_r} = \dfrac{E_r}{\sqrt{r_r^2 + x_r^2}} = \dfrac{SE_2}{\sqrt{r_2^2 + s^2 x_2^2}}$
  - simplifying, $Ir = \dfrac{E_2}{\sqrt{\left(\dfrac{r_2}{S}\right)^2 + X_2^2}}$

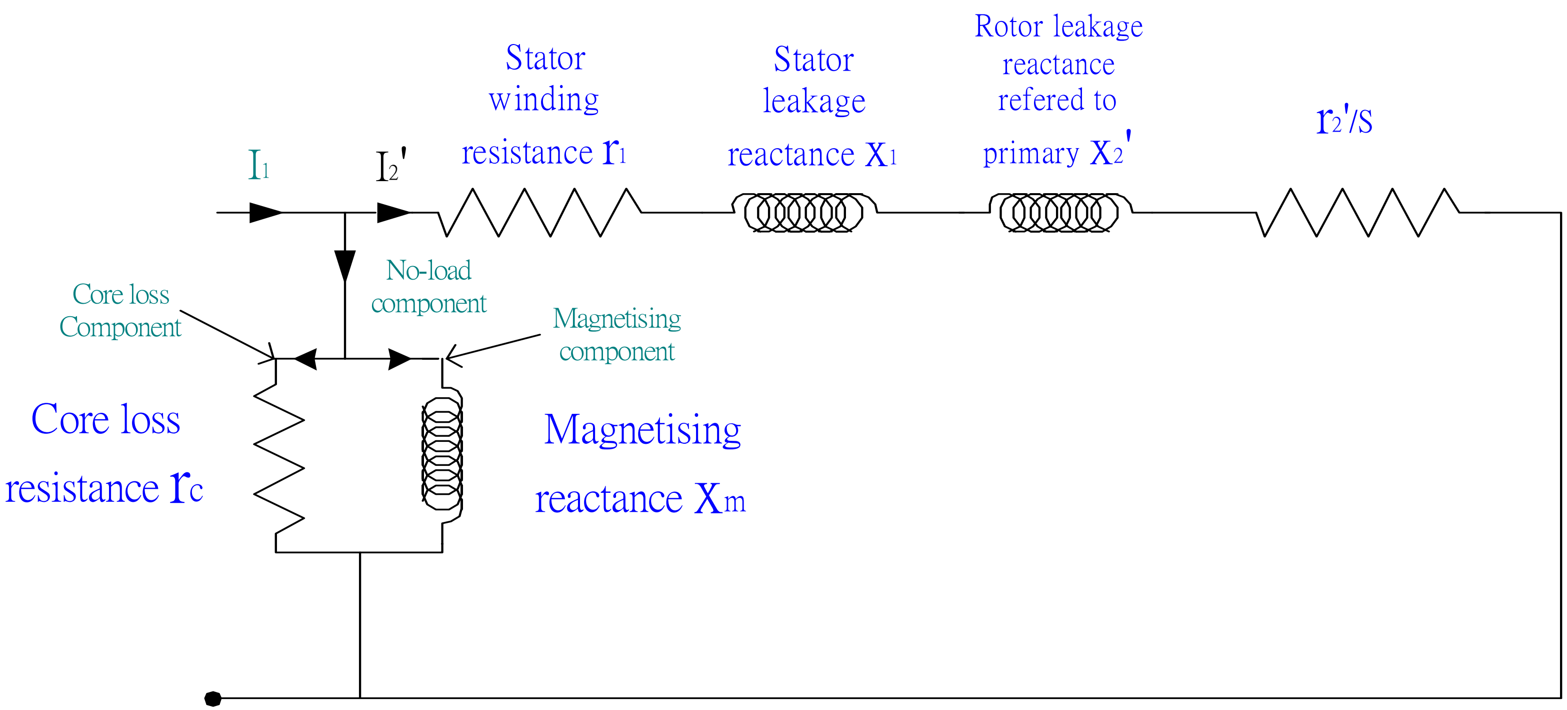
Induction motor equivalent circuit

- Since $I_r = \dfrac{E_2}{\sqrt{\left(\dfrac{r_2}{s}\right)^2 + X_2^{\,2}}}$
- this equation is very similar to the secondary current derived from the transformer equivalent circuit, therefore we see that a running induction motor can be represented by a transformer equivalent circuit except that the rotor resistance has to be revised as $\dfrac{r_2}{s}$



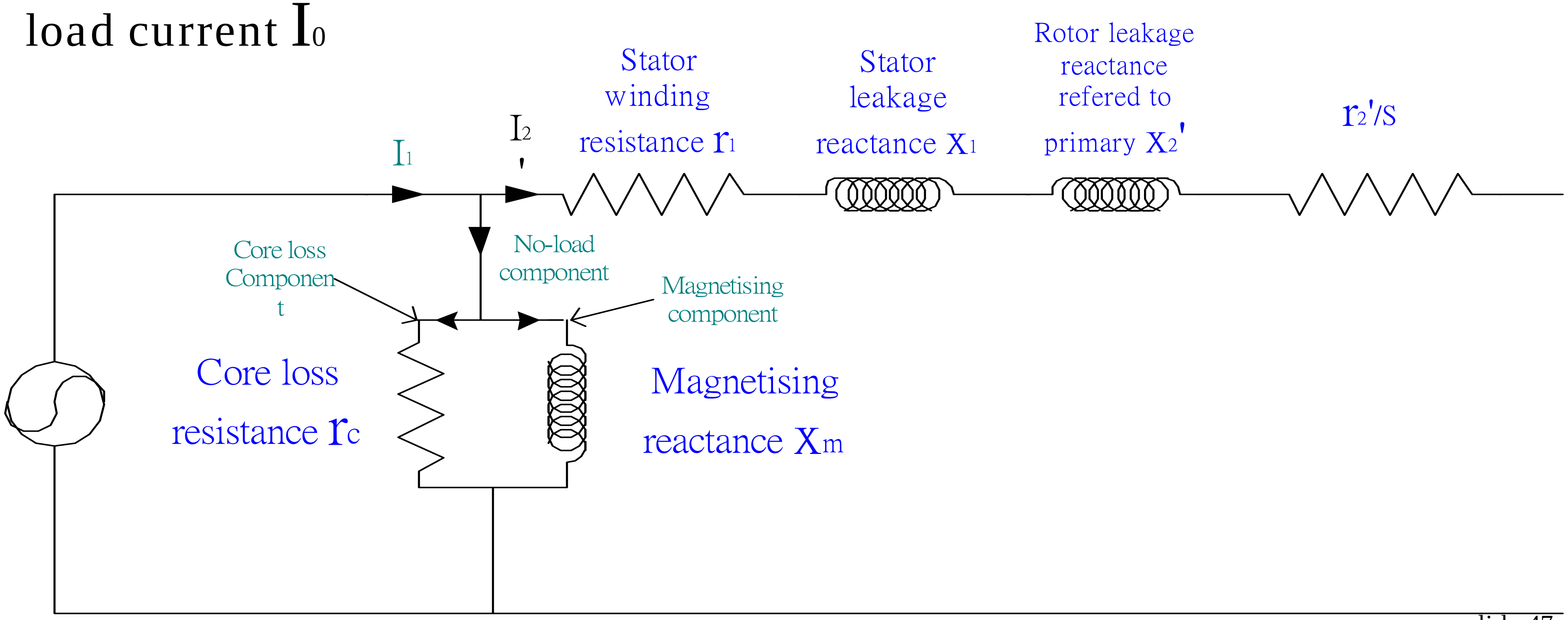
Induction motor approximate equivalent circuit

- Since the No-load component of the equivalent circuit is only approximately 5% of the full load current, we can use the approximate equivalent circuit:



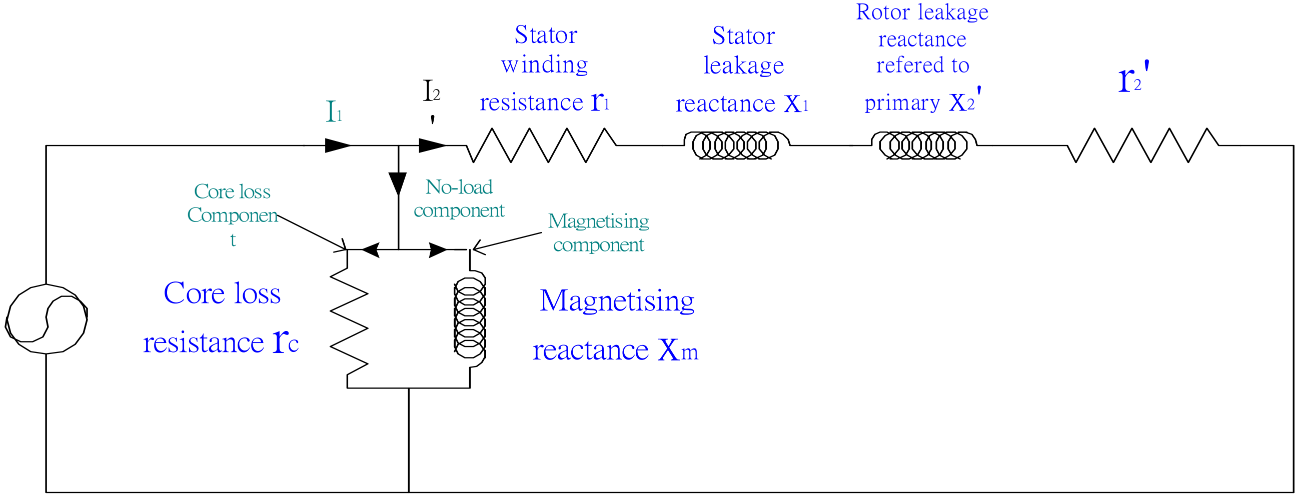
No-load test and clamp rotor test

- Similar to the transformer equivalent circuit, we can use open circuit test and to find out the no-load components $r_c + X_m$
- When an induction motor is unloaded, $S \approx 0$, $\frac{r_2'}{S} \approx \infty$, the secondary circuit is open circuited. All the current taken from the supply is the no load current I_0



No-load test and clamp rotor test

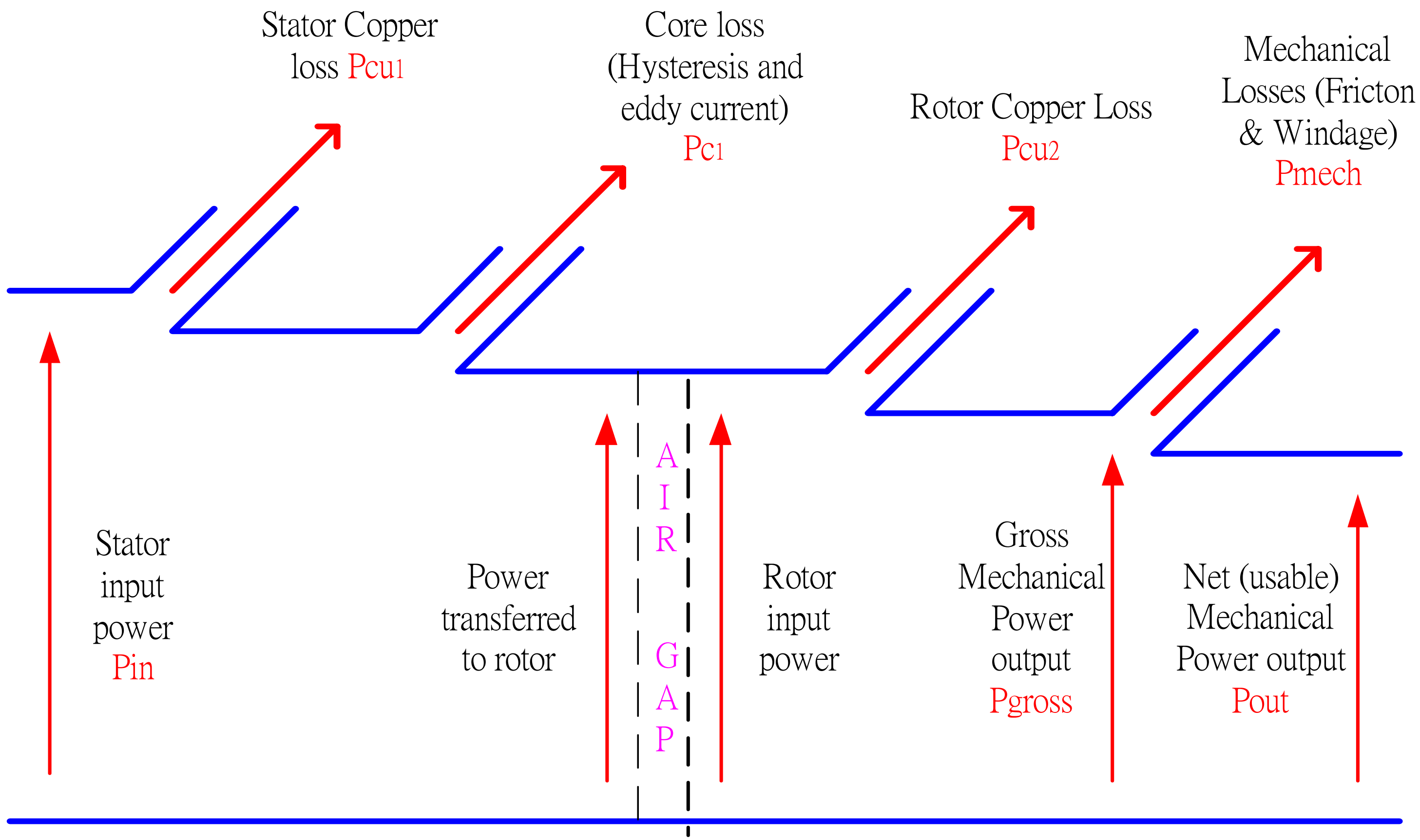
- Similarly, we can use short circuit test to find out the series components, , equivalent resistance $r_1 + r_2'$ and equivalent reactance $X_1 + X_2'$,
- When the rotor of an induction motor is clamped (or locked), $S = 1$, $\frac{r_2'}{S} = r_2'$. Since the secondary circuit is short circuited, we cannot apply full line voltage to the stator. In fact, we only apply reduced voltage (about 1/7) to the stator until full load current flows into the stator. Since No load current is only 1/20 full load current. Reduction of the applied voltage will cause this current to reduces to 1/140 full load current,n from the supply is the no load current I_0



Power and losses of an induction motor

- When the induction motor is connected to a 3 phase supply, current flows in the primary circuit, heat is produced in the stator conductor as copper loss.
- No load current also flows and further power is lost as iron losses
- Power than flow through the air gap as magnetic field energy, there will be no loss in the air gap
- Further energy is lost in the rotor winding as rotor copper loss
- Now the electrical energy is converted to mechanical energy, but before this energy can be fully utilized, further energy is lost as friction and windage losses

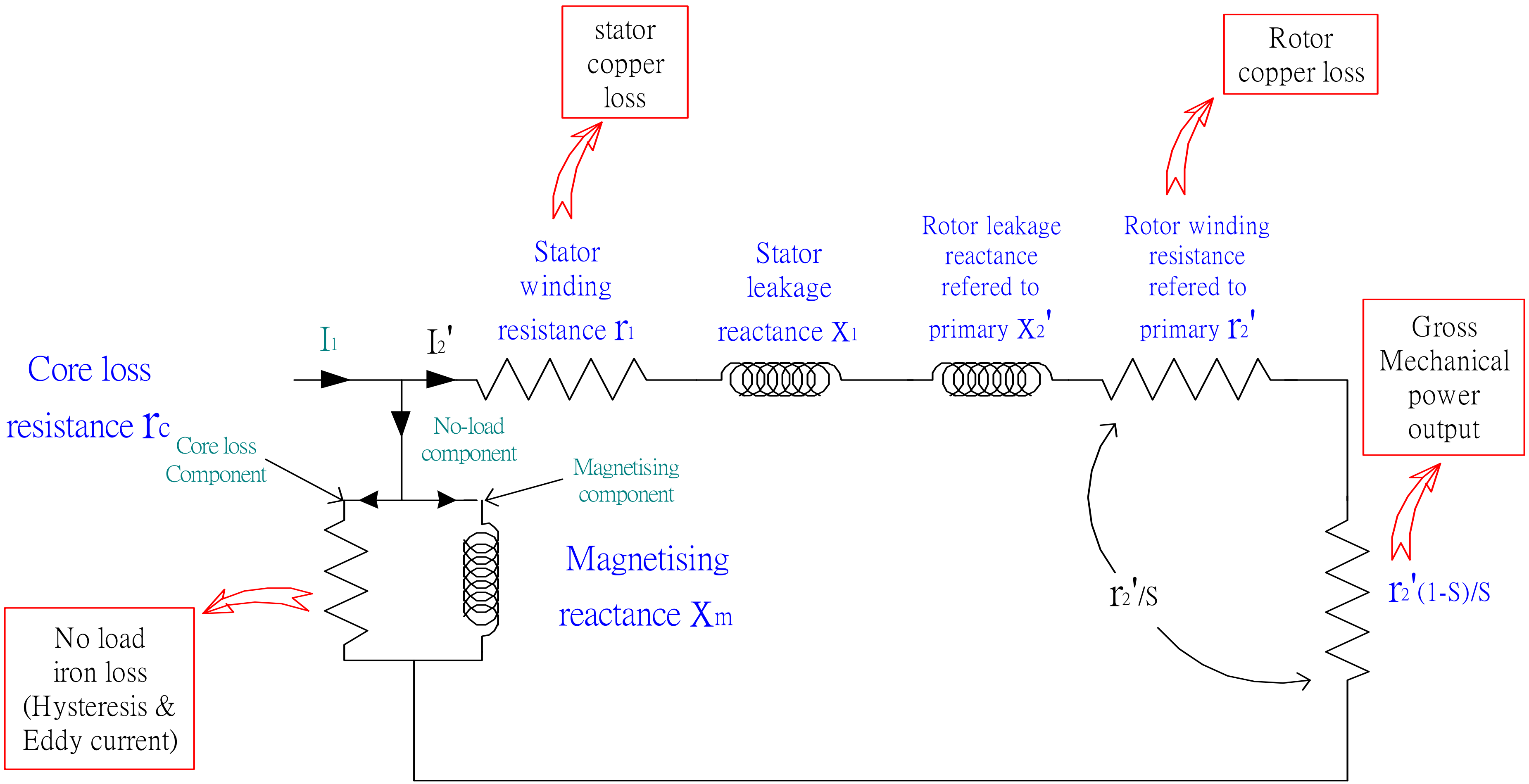
Power and Losses Chart of an induction motor



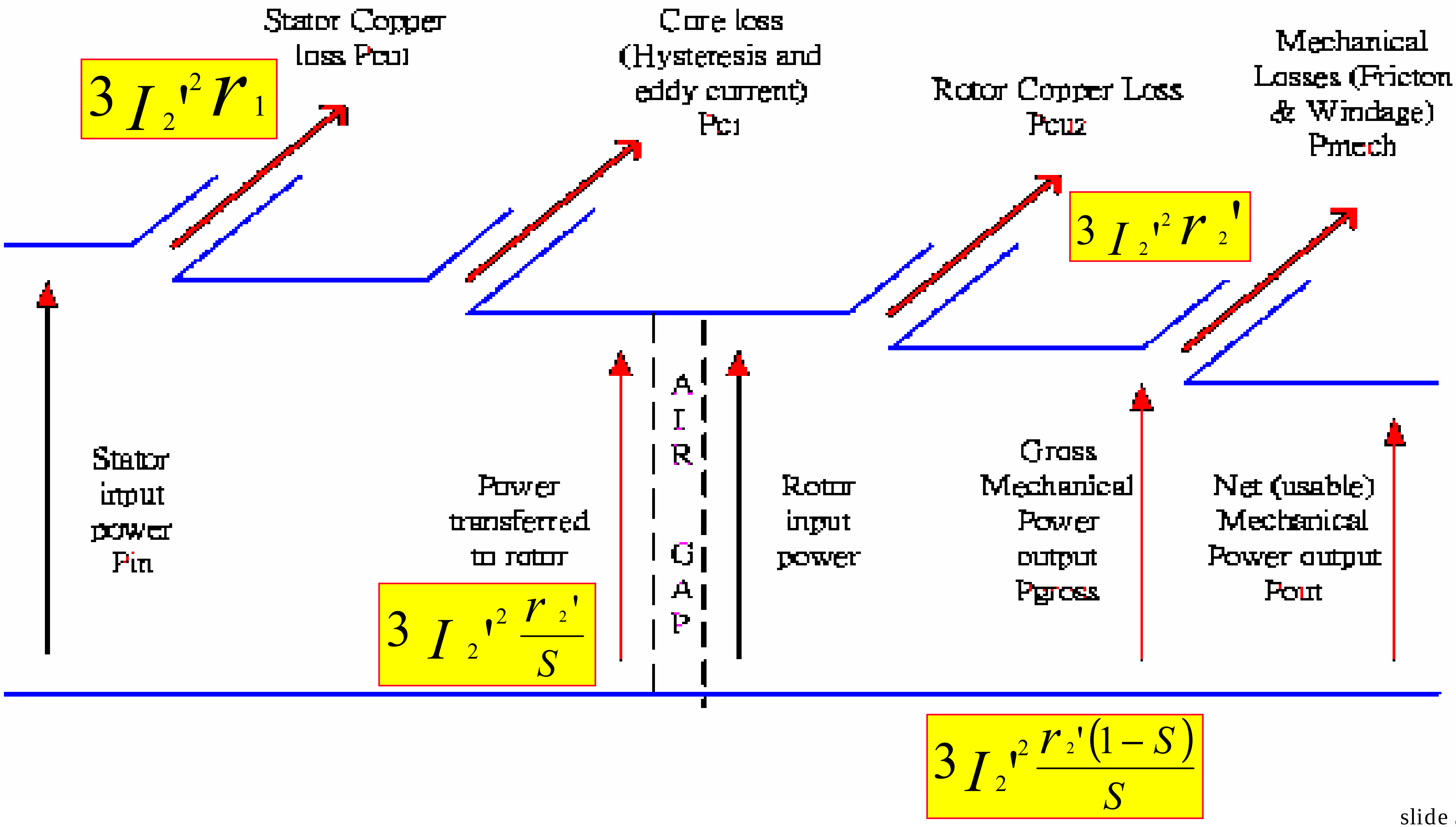
Power representation in the equivalent circuit and its relation to the power flow diagram

- We can relate the power flow diagram and the equivalent circuit's parameters
- Since only resistors will dissipate power, therefore in the induction motor equivalent circuit only resistive components will represent power
- Since a 3-phase induction motor is a balanced load, we can simply use single phase transformer circuit to represent the 3 phase induction motor. However, we must multiply the power in the single phase equivalent circuit by 3 in order to calculate the actual 3-phase power

Power relation in the equivalent circuit



Power representation in power flow chart



Calculation with power and efficiency

- The rating plate of a 3-phase induction motor is as follows:

  Frequency: 50 cycle / second   Power: 20 KW

  Connection: Delta

  Current: 44.6 Amperes   Voltage: 380   Power factor: 0.8

  Revolution: 960 rev / min   Insulation: Class E

- Calculate:
  - A) Efficiency on full load
  - B) Full load torque
  - C) Rotor Copper loss on full load if the mechanical loss = 250Watts
  - D) Core loss if the stator resistance is 0.4 ohm per phase

Calculation with power and efficiency

Solution:

- A) Efficiency $= \dfrac{20\times10^3}{\sqrt{3}\times380\times44.6\times0.8} = 85.2\%$
- B) Full Load torque$= \dfrac{20\times10^3}{2\times\pi\times\frac{960}{60}} = 199\,Newton - meter$
-
- C) Rotor copper loss $= 3\,I_2'^2 r_2'$

Given full load power output = 20 kW, Mechanical loss = 250 W

Gross mechanical power = 20 + 0.25 = 20.25 kW

Gross mechanical output $= 3\,I_2'^2 \dfrac{r_2'(1-S)}{S} = 20.25\times10^3$

Slip S $= \dfrac{Ns - Nr}{Ns} = \dfrac{1000-960}{1000} = 0.04$

Calculation with power and efficiency

Solution Continued:

- C) Rotor copper loss

$$= 3 I_2'^2 r_2' = 3 I_2'^2 r_2' \frac{(1-S)}{S} \times \frac{S}{(1-S)}$$

$$= 20.25 \times 10^3 \times \frac{0.04}{(1-0.04)}$$

$$= 844 \text{ Watts}$$

- D) Total loss of the induction motor $= P_{in} - P_{out} = \frac{P_{out}}{\eta} - P_{out}$

$$= \left(\frac{20}{0.852} - 20 \right) kiloWatts \quad = 3.47 \; kiloWatts$$

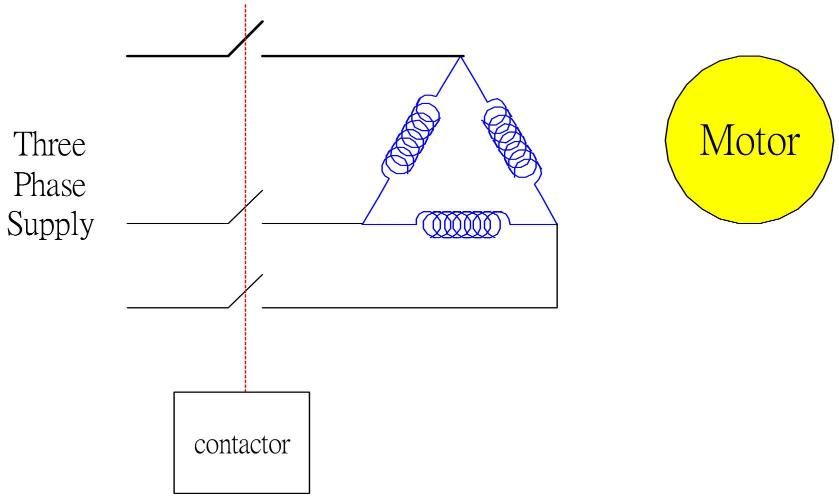
Stator Copper loss $= 3 I_2'^2 r_1 = 3 \times \left(\frac{44.6}{\sqrt{3}} \right)^2 \times 0.4 = 0.796 \, kWatt$

Core loss Pc = Total loss - P_{cu1} - P_{cu2} - P_{mech}

= 3.47 - 0.796 - 0.844 - 0.25 = 1.58 k Watts

Starters for squirrel cage induction motor

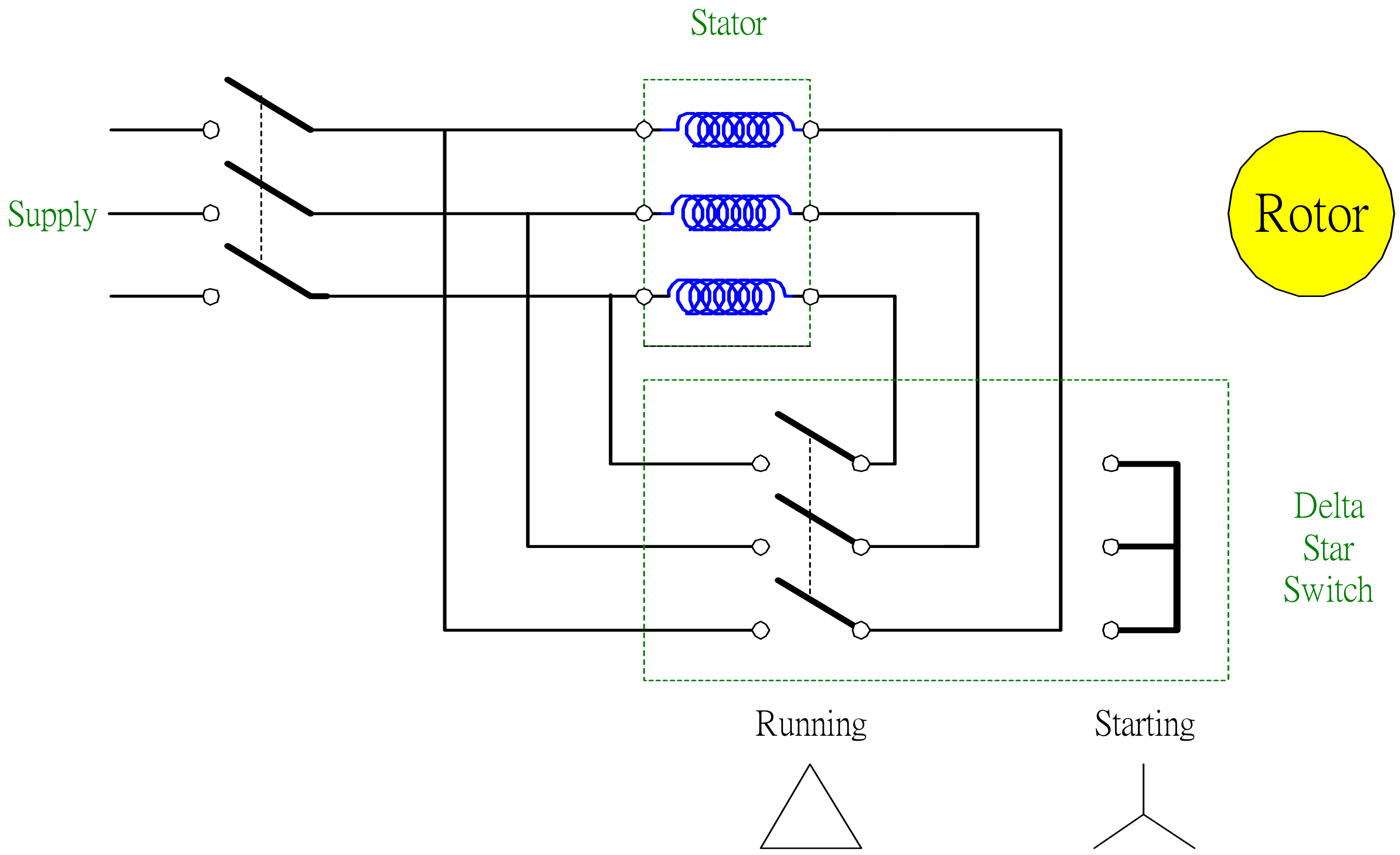
- A squirrel cage motor is at stationary before it is started, there is no back e.m.f. to oppose the current. Therefore, if this motor is connected directly to the supply, will take an initial starting current which is about 5 times the full load value. Though this current decreases rapidly as the motor accelerates, it will cause harm to the motor and will affect the voltage regulation of the power supply
- Small motors up to the size of 5 H.P. are allowed to be started with direct on line (D.O.L.) starter.



Star Delta starter

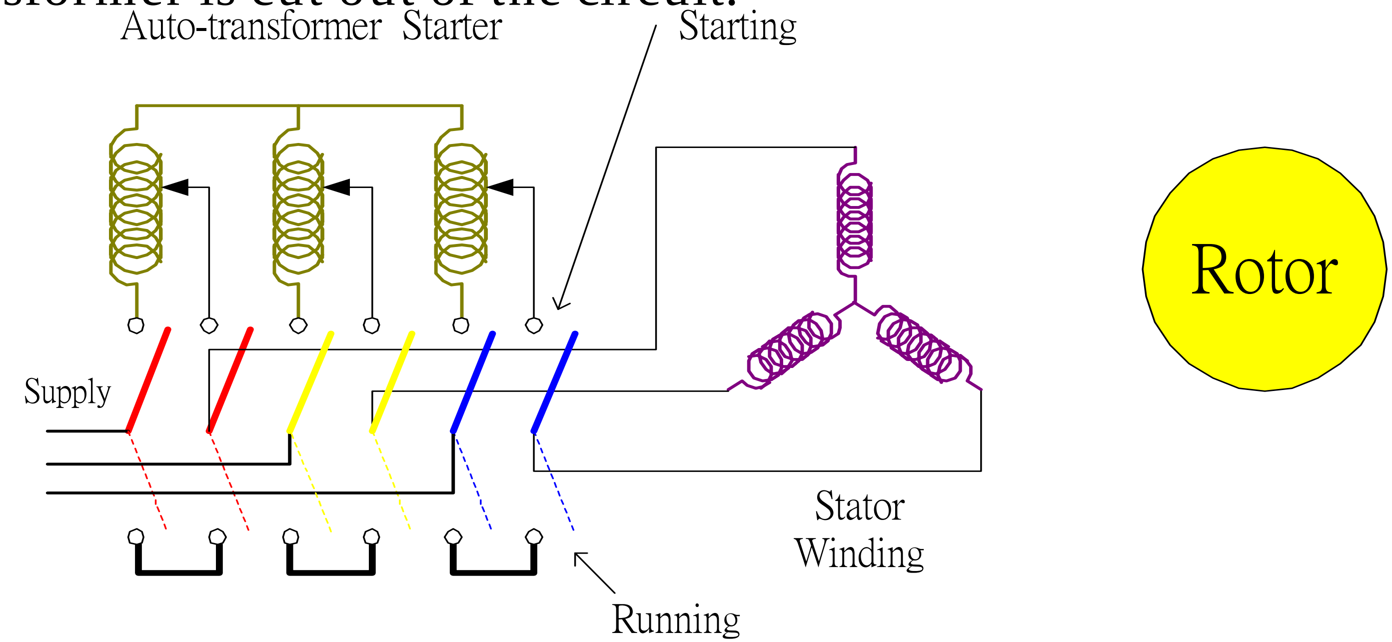
- When the rating of the motor exceed 5 H.P. Some starting means must be used to start the motor. A star / delta starter is normally used because it is the simpliest and cheapest type of starter.
- During starting, the stator winding is temporarily connected in star, therefore only phase voltage is applied to the stator. The starting current is reduced to 1/3 of the Direct on line starting current. The starting torque, which is proportional to the starting current, reduces also to one third of the value at direct on line starting.
- After a period of about 5 seconds, the motor have accelerated to nearly full load speed. The stator winding is now reconnected as delta, and full line voltage is applied each phase of the stator.

Schematic diagram of a star-delta starter



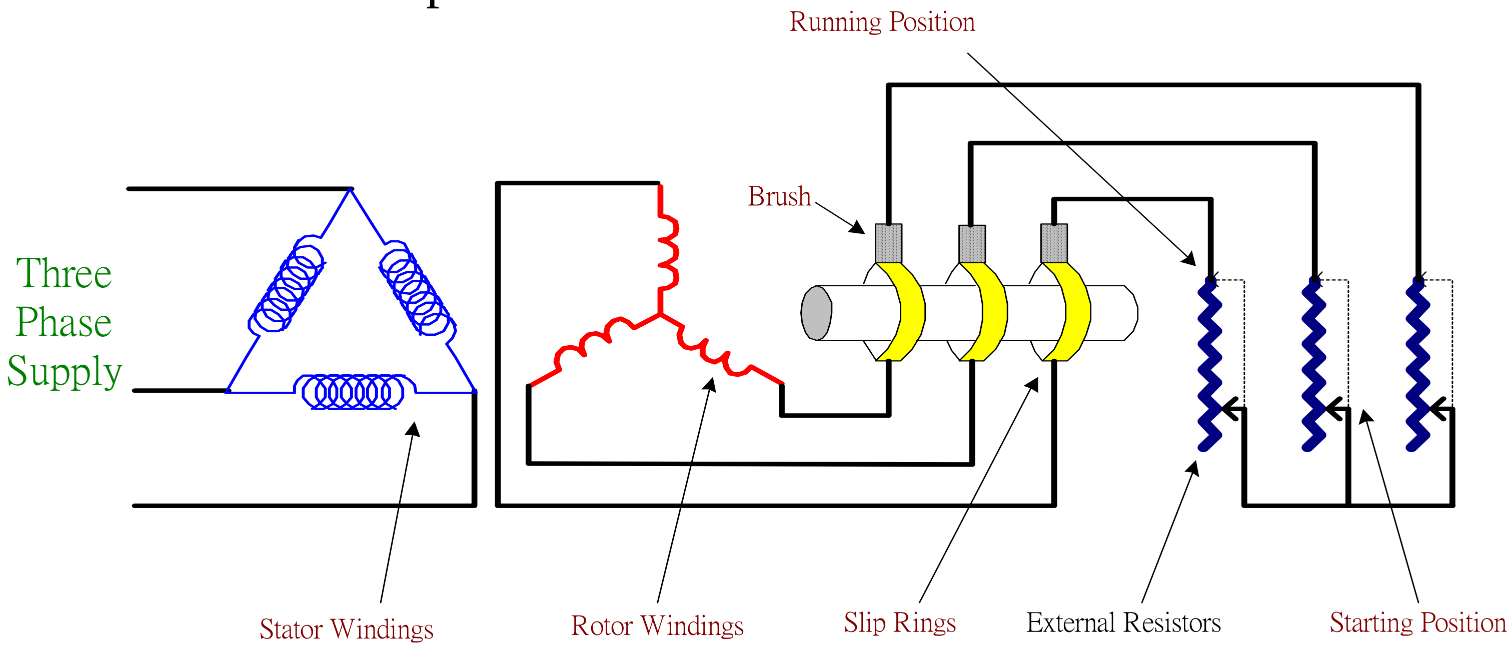
Auto-transformer starter

- Some loads are very heavy and it will take a few minutes before it can run to full speed, these motors have to be started by means of transformer starter. The reduced voltage during starting is obtained from the different tappings (40% , 60% , 75%) of an auto transformer. In the running condition, full voltage is applied to the stator and the transformer is cut out of the circuit.



Starting of Wound (Slip ring) rotor induction motor

- The slip ring induction motor can be started by inserting additional resistance in series with the rotor winding through the slip rings. In this way, maximun torque is obtained during starting. The additional resistance is cut off from the circuit as soon as the motor is started to avoid excessive power loss in the resistance.



Torque Equation

- Gross torque × Angular velocity = Total mechanical power output

$$\Gamma \times \omega_r = 3 \times I_2'^2 \frac{r_2'}{s} - 3 \times I_2'^2 r_2' = 3 I_2'^2 \times r_2' \left(\frac{1-s}{s} \right)$$

But $\omega_r = 2 \times \pi \times \frac{n_1}{60}(1-s)$

$$\Gamma \times 2\pi \frac{n_1(1-s)}{60} = 3 I_2'^2 \times r_2' \left(\frac{1-s}{s} \right)$$

$$\Gamma = \frac{3 I_2'^2 \times r_2' \left(\frac{1-s}{s} \right)}{2\pi \frac{n_1(1-s)}{60}} = \frac{3 I_2'^2 \times \frac{r_2'}{s}}{2\pi \times \frac{f_1 \times P \times 60}{60}} = \frac{3 P I_2'^2 \frac{r_2'}{s}}{2\pi f_1}$$

$$\Gamma = \frac{3 P V_1^2 \frac{r_2'}{s}}{2\pi f_1 \left\{ \left(r_1 + \frac{r_2'}{s} \right)^2 + \left(X_1 + X_2' \right)^2 \right\}}$$

From the torque equatin we note that Torque is a function of slip

Induction motor starting torque

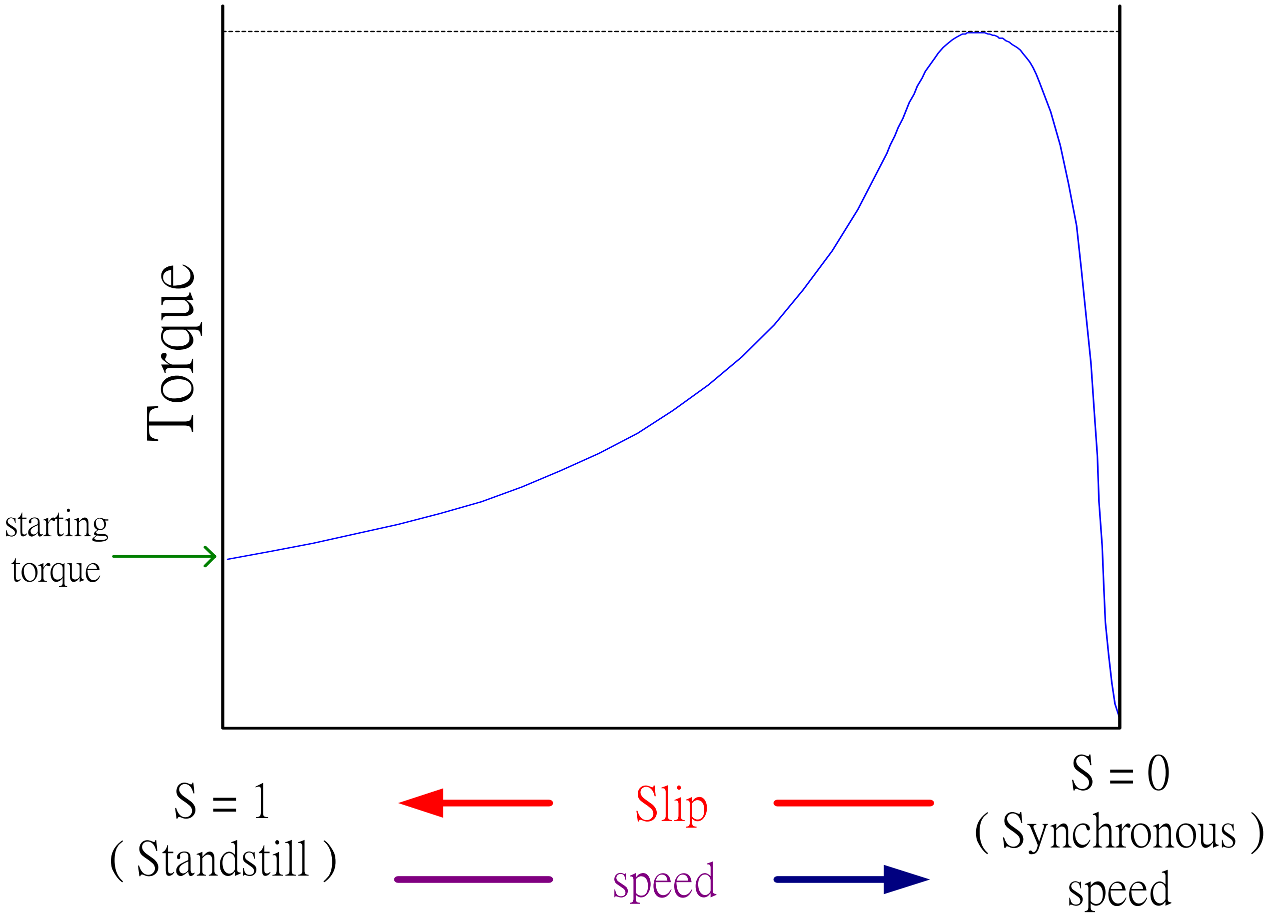
- During starting, the motor speed is zero, the slip S = 1
- since torque

$$\Gamma(s) = \frac{3PV_1^2 \frac{r_2'}{S}}{2\pi f_1 \left\{ \left(r_1 + \frac{r_2'}{S} \right)^2 + \left(X_1 + X_2' \right)^2 \right\}}$$

- starting torque

$$\Gamma_{start} = \frac{3PV_1^2 r_2'}{2\pi f_1 \left((r1 + r2')^2 + (X1 + X2')^2 \right)}$$

Torque / Slip characteristic



Slip at maximun torque

- From the torque equation

$$\Gamma(s) = \frac{3PV_1^2 \dfrac{r_2'}{S}}{2\pi f_1 \left\{ \left(r_1 + \dfrac{r_2'}{S} \right)^2 + \left(X_1 + X_2' \right)^2 \right\}}$$

- We notice that in the torque equation, Torque (y) is the dependent variable while S (x) is the independent variable.
- If we wish to find the value of slip for maximun torque, we have to deferentiate torque (y) with respect to slip (x) and let dy / dx = 0

$$\frac{d\Gamma(s)}{dS} = 0 \Rightarrow \frac{d}{dS} \left(\frac{3PV_1^2 \dfrac{r_2'}{S}}{2\pi f_1 \left\{ \left(r_1 + \dfrac{r_2'}{S} \right)^2 + (X_1 + X_2')^2 \right\}} \right) = 0$$

Slip at maximun torque (Part I)

- Since for a given motor connected to an infinite busbar (ideal voltage source)

$$V_1, f, r_1, r_2', x_1, x_2' = 0$$

then

$$\frac{d\Gamma(s)}{dS} = \frac{d}{dS}\left(\frac{\frac{1}{S}}{\left\{\left(r_1 + \frac{1}{S}\right)^2 + (X_1 + X_2')^2\right\}} \right) = 0$$

$$\frac{dU}{dV} = \frac{UdV - VdU}{V^2} = 0 \Rightarrow V\frac{dU}{dS} = U\frac{dV}{dS}$$

This has the form U / V where

$$\left\{\left(r_1 + \frac{r_2'}{S}\right)^2 + (X_1 + X_2')^2\right\}\left(\frac{-1}{S^2}\right) = \frac{-1}{S} \times 2\left(r_1 + \frac{r_2'}{S}\right)\left(\frac{r_2'}{S^2}\right) \Rightarrow \left(r_1 + \frac{r_2'}{S}\right)^2 + (X_1 + X_2')^2 = \frac{2r_2'}{S}\left(r_1 + \frac{r_2'}{S}\right)$$

$$1 + \frac{(X_1 + X_2')^2}{\left(r_1 + \frac{r_2'}{S}\right)^2} = \frac{2r_2'}{S\left(r_1 + \frac{r_2'}{S}\right)} \Rightarrow \frac{(X_1 + X_2')^2}{\left(r_1 + \frac{r_2'}{S}\right)^2} = \frac{2r_2' - S\left(r_1 + \frac{r_2'}{S}\right)}{S\left(r_1 + \frac{r_2'}{S}\right)}$$

Slip at maximun torque (Part I I)

- Simplifying gives $\dfrac{(X_1+X_2')^2}{\left(r_1+\dfrac{r_2'}{S}\right)^2} = \dfrac{2r_2' - Sr_1 - r_2'}{S} = \dfrac{r_2' - Sr_1}{S} = \dfrac{S\left(\dfrac{r_2'}{S} - r_1\right)}{S}$

$$\frac{(X_1+X_2')^2}{\left(r_1+\dfrac{r_2'}{S}\right)^2} = \left(\frac{r_2'}{S} - r_1\right) \quad \Rightarrow \quad \frac{(X_1+X_2')^2}{\left(r_1+\dfrac{r_2'}{S}\right)^2} = \left(\frac{r_2'}{S} - r_1\right)$$

$$(X_1+X_2')^2 = \left(\frac{r_2'}{S}\right)^2 - r_1^2 \quad \Rightarrow \left(\frac{r_2'}{S}\right)^2 = r_1^2 + (X_1+X_2')^2$$

$$\frac{r_2'}{S} = \sqrt{r_1^2 + (X_1+X_2')^2} \qquad \Rightarrow S_{\max\Gamma} = \frac{r_2'}{\sqrt{r_1^2 + (X_1+X_2')^2}}$$

Maximun torque & its characteristics

$$\Gamma_{(s)} = \frac{3PV_1^2 \frac{r_2'}{s}}{2\pi f_1 \left\{ \left(r_1 + \frac{r_2'}{s} \right)^2 + \left(X_1 + X_2' \right)^2 \right\}}$$

$$\frac{r_2'}{S_{\max\Gamma}} = \sqrt{r_1^2 + \left(X_1 + X_2' \right)^2}$$

$$\Gamma_{\max} = \frac{3PV_1^2 \frac{r_2'}{S_{\max}}}{2\pi f_1 \left\{ \left(r_1 + \frac{r_2'}{S_{\max\Gamma}} \right)^2 + \left(X_1 + X_2' \right)^2 \right\}}$$

$$\Rightarrow \Gamma_{\max} = \frac{3PV_1^2 \sqrt{r_1^2 + \left(x_1 + x_2' \right)^2}}{2\pi f_1 \left\{ \left(r_1 + \sqrt{r_1^2 + \left(x_1 + x_2' \right)^2} \right)^2 + \left(x_1 + x_2' \right)^2 \right\}}$$

Note that the maximum torque do not depend on the value of rotor resistance, but the value of rotor resistance will affect at what slip (speed) the maximum torque will occur.

Effect of additional rotor resistance on starting torque